STAT 2630

Day 1: 1/7/20

1.1 Terminology

- A **random experiment** is a repeatable process that results in a random *outcome*.
- The collection of all possible outcomes is called the **sample space**, and is typically denoted $\mathcal{C}$.

Example 1.1.2 Roll one red and one white die. This is a random experiment with sample space

$$\mathcal{C} = \{(1,1),(1,2),\ldots,(1,6),(2,1),(2,2),\ldots,(2,6),\ldots,(6,1),(6,2),\ldots,(6,6)\}$$

or more conveniently

		White					
		1	2	3	4	5	6
	1						
	2						
Red	3						
	4						
	5						
	6						

Example 1.1.3 Roll one red and one white die and let B denote the event that the two dice sum to 7. Then B is an **event**: $B = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$. Note that any event B is a subset of $\mathcal{C}$.

		White					
		1	2	3	4	5	6
	1						✓
	2					✓	
Red	3				✓		
	4			✓			
	5		✓				
	6	✓					

The **relative frequency** by which an event B occurs is called its **probability**.

Example 1.1.3b Roll one red and one white die and take the sum of the two dice. This is a random experiment with sample space

$$\mathcal{C} = \{2,3,4,\ldots,11,12\}$$

1.2.1 Review of Set Theory

Definitions:

1. The **complement** of an event A, denoted A^c, is the set of all elements in $\mathcal{C}$ which are not in A.

2. If each element in A is also in B then A is a **subset** of B. This is denoted $A \subset B$.

3. The **union** of A and B is the set of all elements that are in A or in B or in both. This is denoted $A \cup B$, and represents the event that "A or B occurs".

4. The **intersection** of A and B is the set of all elements that are in both A and B. This is denoted $A \cap B$, and represents the event that "A and B occur".

5. DeMorgan's Laws
$$(A \cap B)^c = A^c \cup B^c$$
$$(A \cup B)^c = A^c \cap B^c$$

Example 1.2.1 (p. 5)

1.3 Probability

Definition 1.3.1. *A* **probability function** P *satisfies the following:*

1. *$P(A) \geq 0$ for all events $A \subset \mathcal{C}$.*

2. *$P(\mathcal{C}) = 1$*

3. *If A_n is a mutually exclusive sequence of events (i.e. $A_m \cap A_n = \phi$ for all $m \neq n$), then*
$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

Theorems (p.13)

1. $P(A) = 1 - P(A^c)$

2. $P(\phi) = 0$

3. If $A \subset B$, then $P(A) \leq P(B)$

4. $0 \leq P(A) \leq 1$

5. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Definition 1.3.2. *Let $\mathcal{C} = \{x_1, x_2, \ldots, x_m\}$ be a finite sample space. If $x_1, \ldots, x_m$ are equally likely, then*

1. *$P(x_i) = 1/m$ for all $i = 1, 2, \ldots, m$ and*
2. *$P(A) = \frac{\#(A)}{m}$ for all events A.*

where $\#(A)$ denotes the number of elements in A.

1.3.1 Counting Rules

Three counting rules

1. Multiplication Rule

   If there are m ways to choose the first element of an ordered pair and there are n ways to choose the second element, then there are mn possible ordered pairs.

   (Extension: If there are m_1 ways to choose the first element, ..., and m_k ways to choose the kth element, then there are $(m_1)(m_2)\cdots(m_k)$ possible k-tuples.

   Examples

   (a) How many license plates can we make consisting of one letter of the alphabet followed by a digit between 0 and 9?
   Ans: (26)(10)=260.

   (b) How many license plates can we make consisting of three letters of the alphabet followed by four digits between 0 and 9?
   Ans: (26)(26)(26)(10)(10)(10)(10)=4,569,760,000

2. Permutations

   Let A be a set with n elements. How many k-tuples can be made from elements of A? *Ans:* $n \cdot n \cdots n = n^k$. How many k-tuples can be made if each component is distinct(i.e. no repeats)? *Ans:* $n(n-1)\cdots(n-(k-1))$. We call such k-tuple a **permutation**. The number of distinct permutations of k objects from n is

   $$P_k^n = n(n-1)\cdots(n-(k-1)) = \frac{n!}{(n-k)!} \tag{1.3.5}$$

   Example 1.3.3 (Birthday Problem) There are 10 people in a room. Assuming that birthdays are equally likely to fall on any of the 365 days of the year, what is the probability that at least 2 people have the same birthday?

   Ans: Represent each set of birthdays by a 10-tuple of integers between 1 and 365. There are a total of $(365)^{10}$ possible outcomes. Let A be the subset where no two integers match. There are $P_{10}^{365} = 365(364)\cdots(356)$ of these. Then

   $$P[\text{Matching birthdays}] = 1 - P[\text{No match}] = 1 - \frac{P_{10}^{365}}{(365)^{10}}$$

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> # COMPUTING IN R
> bday=function(n){bday=1; nm1=n-1
+                   for(j in 1:nm1){bday=bday*((365-j)/365)}
+                                  bday<-1-bday
+                                  return(bday)}
> bday(10)
[1] 0.1169482
> for(i in 10:25){a[i]<-bday(i)}
> a[10:25]
 [1] 0.1169482 0.1411414 0.1670248 0.1944103 0.2231025
 [6] 0.2529013 0.2836040 0.3150077 0.3469114 0.3791185
[11] 0.4114384 0.4436883 0.4756953 0.5072972 0.5383443
[16] 0.5686997
```

3. Combinations

   A poker hand consists of 5 randomly drawn cards drawn *without repetition* from a regular deck of 52 playing cards. How many different poker hands are there? This is called a **combination** of 5 taken from 52. The difference between a permutation and combination is that permutations are ordered k-tuples, i.e. the order matters. Pretending for now that order matters, there are P_5^{52} 5-tuples of cards from 52. However, each unordered poker hand corresponds to 5! ordered 5-tuples. So

$$P_5^{52} = 5! C_5^{52}$$

   where C_5^{52} is the number of unordered sets of 5 cards from 52. In general, if we draw k objects from n, the number of combinations is

$$C_k^n \equiv \binom{n}{k} = \frac{P_k^n}{k!} = \frac{n!}{k!(n-k)!} \tag{1.3.6}$$

Example 1.3.4 (Poker hands)

1. Given a 5-card poker hand, what is the probability of a flush?

$$P[\text{flush}] = \frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}}$$

2. What is the probability of getting three-of-a-kind?

$$P[\text{three of a kind}] = \frac{\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2}{\binom{52}{5}}$$