STAT 2630

Day 2: 1/9/20

1.4 Conditional Probability and Independence

Definition 1.4.1. *Let B and A be events with $P(A) > 0$. The* **conditional probability** *of B given A is*

$$P(B|A) = \frac{P(A \cap B)}{P(A)}. \tag{1.4.1}$$

Example 1.4.1 A five-card poker hand is known to have at least four spades. Find the probability that all five cards are spades.

Ans:

$$P(\text{all spades}|\text{at least 4 spades}) = \frac{P(\text{at least 4 spades} \cap \text{all spades})}{P(\text{at least 4 spades})} = \frac{P(\text{all spades})}{P(\text{at least 4 spades})}$$

$$= \frac{\binom{13}{5}/\binom{52}{5}}{\binom{13}{4}\binom{39}{1}/\binom{52}{5} + \binom{13}{5}/\binom{52}{5}} = \frac{\binom{13}{5}}{\binom{13}{4}\binom{39}{1} + \binom{13}{5}} = .0441$$

Rearranging terms in equation (1.4.1), we get the **probability multiplication rule**

$$P(A \cap B) = P(A)\ P(B|A) \tag{1.4.1b}$$

which is useful for computing probabilities of multi-stage events.

Example 1.4.2 A bowl contains three red chips and five blue chips. Two chips are drawn in succession and without replacement.

1. Find the probability that the first chip is red, and the second is blue.

   Ans: $P(\text{first is red and second is blue}) \equiv P(R_1 \cap B_2) = P(R_1)P(B_2|R_1) = \left(\frac{3}{8}\right)\left(\frac{5}{7}\right)$

2. Find the probability that the two draws are different colors.

   Ans: $P(\text{different colors}) = P(B_1R_2\ \cap\ R_1B_2) = \left(\frac{3}{8}\right)\left(\frac{5}{7}\right) + \left(\frac{5}{8}\right)\left(\frac{3}{7}\right)$

The probability multiplication rule can be extended to more than two events.

$$P(A \cap B \cap C) = P(A)P(B|A)P(C|A \cap B)$$

Exer. 1.3.10 (a) A bowl contains 6 red chips, 7 white chips, and 3 blue chips. If four chips are drawn, find the probability that all chips are red.

Ans: $P(R_1R_2R_3R_4) = P(R_1)P(R_2|R_1)P(R_3|R_1R_2)P(R_4|R_1R_2R_3) = \left(\frac{6}{16}\right)\left(\frac{5}{15}\right)\left(\frac{4}{14}\right)\left(\frac{3}{13}\right)$

Two-stage events

Example 1.4.5 Bowl A_1 contains three red and seven blue chips. Bowl A_2 contains eight red and two blue chips. A die is rolled and a chip is randomly selected from bowl A_1 if the die shows 5 or 6; otherwise a chip is randomly selected from bowl A_2.

1. What is the probability of the event B that a red chip is drawn?

   Solution:

   The two bowls have **prior probabilities** $P(A_1) = 2/6$ and $P(A_2) = 4/6$. Furthermore, we have the following conditional probabilities: $P(B|A_1) = 3/10$ and $P(B|A_2) = 8/10$. Now

   $$\begin{aligned} P(B) &= P((B \cap A_1) \cup (B \cap A_2)) \\ &= P(B \cap A_1) + P(B \cap A_2) \\ &= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) \\ &= \left(\frac{3}{10}\right)\left(\frac{2}{6}\right) + \left(\frac{8}{10}\right)\left(\frac{4}{6}\right) = \frac{19}{30} \end{aligned}$$

   Note that $P(B)$ is a weighted average of $P(B|A_1)$ and $P(B|A_2)$, using the priors as weights.

   $$P(B) = \frac{\frac{3}{10} + \frac{3}{10} + \frac{8}{10} + \frac{8}{10} + \frac{8}{10} + \frac{8}{10}}{6} = \frac{19}{30}$$

   In general, let $A_1, A_2, \ldots, A_k$ be a partition of $\mathcal{C}$ (i.e. $\{A_i\}$ are mutually exclusive and exhaustive). Then

   $$\begin{aligned} P(B) &= P(B \cap A_1) + P(B \cap A_2)P(B|A_1)P(A_1) + \cdots + P(B \cap A_k) \\ &= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_k)P(A_k) \\ &= \sum_{i=1}^{k} P(B|A_i)P(A_i) \end{aligned}$$

   This result is called the **law of total probabilities**.

2. Suppose a red chip was drawn in stage 2. What is the probability it came from bowl A_1?

   Solution:

   $$\begin{aligned} P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} &= \frac{P(B|A_1)P(A_1)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2)} \\ &= \frac{\left(\frac{3}{10}\right)\left(\frac{2}{6}\right)}{\left(\frac{3}{10}\right)\left(\frac{2}{6}\right) + \left(\frac{8}{10}\right)\left(\frac{4}{6}\right)} = \frac{3}{19} \end{aligned}$$

   Similarly, $P(A_2|B) = \frac{16}{19}$.

Theorem 1.4.1. *Let $A_1, A_2, \ldots, A_k$ be a partition of $\mathcal{C}$. Let B be any event. Then*

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$$

Example 1.4.6 (p.27)
Example 1.4.7 (p.28) The False Positive problem in random testing

1.4.1 Independence

Sometimes, the occurrence of event A does not change the probability of event B, i.e. $P(B|A) = P(B)$. By the probability multiplication rule, $P(A \cap B) = P(A)P(B|A) = P(A)P(B)$.

Definition 1.4.2. *Two events A and B are* **independent** *if $P(A \cap B) = P(A)P(B)$. More generally, the n events $A_1, A_2, \ldots, A_n$ are* **mutually independent** *if and only if*

$$P(A_{d_1} \cap A_{d_2} \cap \cdots \cap A_{d_k}) = P(A_{d_1})P(A_{d_2}) \cdots P(A_{d_k})$$

for all subsets of k of these events, $2 \leq k \leq n$.

Independent events often arise when we conduct *independent experiments*. For example, when we roll a die n times, then it is reasonable to assume that the outcome of one roll does not affect the probability of outcome of the other rolls.

Example 1.4.10 A coin is flipped four times. Find the probability of

1. the ordered sequence HHTH

   Sol'n: $P(HHTH) = P(H \cap H \cap T \cap H) = P(H)P(H)P(T)P(H) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$

2. the first head appears on the third flip

   Sol'n: $P(TTHH \cup TTHT) = P(TTHH) + P(TTHT) = \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4 = \frac{1}{8}$

3. at least one head

   Sol'n: The complement of "at least one" is "none", so

$$P(\text{at least one H}) = 1 - P(TTTT) = 1 - \left(\frac{1}{2}\right)^4 = \frac{15}{16}$$

Example 1.4.11 A computer system is built with back ups in place so that if component K_1 fails, then K_2 is used and if K_2 fails, then K_3 is used. If the failure probability of K_1 is .01, of K_2 is .03, and of K_3 is .08, how relaible is the system? (Calculate the probability that the system does not fail.)

Sol'n:

$$\begin{aligned} P(\text{system works}) &= 1 - P(\text{all 3 components fail}) \\ &= 1 - (.01)(.03)(.08) \\ &= 1 - .00002 \\ &= .999976 \end{aligned}$$