Random Variables

Day 3 (1/14/20)

1.5 Random variables

Example 1.5.1 Roll a pair of dice. Let X be the sum of the two dice. Then X has the **probability mass function**

x	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(1)

This is an example of a **discrete** random variable because the **support** of X (i.e. the range of possible values) is a countable set. The pmf allows the calculation of probabilities, for example

$$P(\text{craps}) = P(2 \text{ or } 3 \text{ or } 12) = p(2) + p(3) + p(12) = \frac{1}{36} + \frac{2}{36} + \frac{1}{36} = \frac{4}{36}$$

A discrete random variable may have an infinite number of points in its support. For example, roll a die repeatedly and let X be the roll number on which the first '6' occurs. Then X has pmf

x	1	2	3	4	5	$\ldots$
$p(x)$	$\frac{1}{6}$	$\left(\frac{5}{6}\right)\frac{1}{6}$	$\left(\frac{5}{6}\right)^2\frac{1}{6}$	$\left(\frac{5}{6}\right)^3\frac{1}{6}$	$\left(\frac{5}{6}\right)^4\frac{1}{6}$	$\ldots$

or more compactly

$$p(x) = \left(\frac{5}{6}\right)^{x-1}\frac{1}{6} \quad \text{for} \quad x = 1, 2, 3, 4, \ldots$$

A **continuous random variable** has a support that is a continuous interval. For example, let X be the time in seconds between calls to the customer support center of an online retailer. Then X takes values on the interval $(0, \infty)$. Instead of a table like (1), probabilities for continuous random variable are represented by an integrable **probability density function** $f(\cdot)$ such that

$$P[a < X < b] = \int_a^b f(x)dx \tag{2}$$

For example, suppose that

$$f(x) = \begin{cases} \frac{1}{4}e^{-x/4} & 0 < x < \infty \\ 0 & \text{elsewhere} \end{cases}$$

What percentage of between-call times exceed 4 seconds?

$$P[X > 4] = \int_4^\infty \frac{1}{4}e^{-x/4}dx = -e^{-x/4}|_4^\infty = -(0 - e^{-1}) = e^{-1} = .3679$$

Because the pdf satisfies (2), then individual points have 0 probability when a random variable is continuous, for example

$$P[x = 4] = \int_4^4 \frac{1}{4}e^{-x/4}dx = 0$$

so that (2) may be written as

$$P[a < X < b] = P[a \le X \le b] = \int_a^b f(x)dx$$

For either discrete or continuous random variables, the **cumulative distribution function** is

$$F(x) = P[X \le x]$$

Example 1.5.3 (discrete)

Example 1.5.4 (continuous)

1.7.2 Transformations

Suppose we know the distribution of X. Can we find the distribution of some transformation $Y = g(X)$?

Case 1: Discrete

For discrete X, the problem is not hard. For example, let X be the outcome of a die roll so that X has pmf

x	1	2	3	4	5	6
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

What is the pmf of $Y = (X-3)^2$? If $x = 1$ then $y = (1-3)^2 = 4$. Continuing with other values

y	4	1	0	1	4	9
$p(y)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

so that Y has pmf

y	0	1	4	9
$p(y)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

Case 2: Continuous

Let X have pdf

$$f(x) = \begin{cases} \frac{1}{4}e^{-x/4} & 0 < x < \infty \\ 0 & \text{elsewhere} \end{cases}$$

What is the pdf, say $g(\cdot)$, of $Y = X^2$? It is not easy to find $g(\cdot)$ directly, because the pdf is *not* a probability. So we look for g in two steps.

- Step 1: Calculate the cdf of Y

$$G(y) = P(Y \le y) = P(X^2 \le y) = P(X \le \sqrt{y}) = \int_0^{\sqrt{y}} f(x)dx$$

- Step 2: Calculate the pdf $g(y)$ from $G(y)$. The main tool here is the Fundamental Theorem of Calculus

$$F(x) = \int_{-\infty}^{x} f(t)dt \text{ implies } \frac{d}{dx}F(x) = f(x) \tag{1.7.2}$$

Solution: For $y > 0$,

$$\begin{aligned} G(y) = P(Y \le y) = P(X^2 \le y) = P(X \le \sqrt{y}) &= \int_0^{\sqrt{y}} f(x)dx \\ = \int_0^{\sqrt{y}} \frac{1}{4}e^{-x/4}dx = -e^{-x/4}|_0^{\sqrt{y}} &= 1 - e^{-\sqrt{y}/4} \end{aligned}$$

Then

$$\begin{aligned} g(y) &= \frac{d}{dy}G(y) = \frac{d}{dy}\left(1 - e^{-\sqrt{y}/4}\right) \\ &= -e^{-\sqrt{y}/4}\frac{d}{dy}(-\sqrt{y}/4) \\ &= \frac{1}{8}y^{-1/2}e^{-\sqrt{y}/4} \end{aligned}$$

so that $Y = X^2$ has pdf

$$g(y) = \begin{cases} \frac{1}{8}y^{-1/2}e^{-\sqrt{y}/4} & 0 < y < \infty \\ 0 & \text{elsewhere} \end{cases}$$

1.8 Expectation

Definition 1.8.1. *The* **expected value** *or* **mean** *of a random variable X is*

$$E(X) = \begin{cases} \sum_x x\, p(x) & \textit{if discrete} \\ \int_x xf(x)dx & \textit{if continuous} \end{cases}$$

Example 1.8.1
Example 1.8.2
Example 1.8.3

Theorem 1.8.1. *The expected value of a function $g(X)$ is*

$$E(X) = \begin{cases} \sum_x g(x)\, p(x) & \textit{if discrete} \\ \int_x g(x)f(x)dx & \textit{if continuous} \end{cases}$$

Example 1.8.4
Example 1.8.5
Example 1.8.6
Example 1.8.7
Example 1.8.8