The Mean and Variance

Day 4 (1/16/20)

1.9 Some Special Expectations

Definition 1.9.1. *The* **mean** *of a random variable* X *is* $\mu = E(X)$.

Definition 1.9.2. *The* **variance** *of a random variable* X *is* $\sigma^2 = E(X-\mu)^2$.

Note that

$$\begin{aligned}\sigma^2 &= E(X-\mu)^2 = E(X^2 - 2\mu X + \mu^2) = E(X^2) - E(2\mu X) + E(\mu^2)\\ &= E(X^2) - 2\mu^2 + \mu^2 = E(X^2) - \mu^2\end{aligned}$$

which is the easier to compute. The square root of the variance is called the **standard deviation**.

$$\sigma = \sqrt{E(X-\mu)^2}$$

Theorem 1.9.1. *Let* a *and* b *be constants. Then*

$$Var(aX+b) = a^2\, Var(X)$$

Proof.

$$\text{Var}(aX+b) = E\left[(aX+b) - (a\mu+b)\right]^2 = E\left[a(X-\mu)\right]^2 = a^2 E\left(X-\mu\right)^2$$

□

Ex. Let X have pmf

x	2	4	5	6	8
$p(x)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

.

1. Find the mean, variance and standard deviation.

   Solution:

$$\begin{aligned}\mu &= E(X) = \sum xp(x)\\ &= 2\left(\frac{1}{5}\right) + 4\left(\frac{1}{5}\right) + 5\left(\frac{1}{5}\right) + 6\left(\frac{1}{5}\right) + 8\left(\frac{1}{5}\right) = 5.0\end{aligned}$$

$$\begin{aligned}\sigma^2 &= E(X-\mu)^2\\ &= (2-5)^2\left(\frac{1}{5}\right) + (4-5)^2\left(\frac{1}{5}\right) + (5-5)^2\left(\frac{1}{5}\right) + (6-5)^2\left(\frac{1}{5}\right) + (8-5)^2\left(\frac{1}{5}\right)\\ &= (9+1+0+1+9)\left(\frac{1}{5}\right) = 4.0\end{aligned}$$

$$\sigma = \sqrt{4.0} = 2.0$$

   The standard deviation is (an approximation of) "average size of deviation from the mean".

2. Let $Y = 3X + 5$. Find the mean, variance and standard deviation.

   (a) *Solution 1:* (Find pmf of $Y = 3X + 5$)

   Y has pmf

y	11	17	20	23	29
$p(y)$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

   $$\mu_Y = E(Y) = \sum y\, p(y)$$
   $$= 11\left(\frac{1}{5}\right) + 17\left(\frac{1}{5}\right) + 20\left(\frac{1}{5}\right) + 23\left(\frac{1}{5}\right) + 29\left(\frac{1}{5}\right) = 20.0$$
   $$\sigma_Y^2 = E(Y - \mu_Y)^2$$
   $$= (11-20)^2\left(\frac{1}{5}\right) + (17-20)^2\left(\frac{1}{5}\right) + (20-20)^2\left(\frac{1}{5}\right) + (23-20)^2\left(\frac{1}{5}\right) + (29-20)^2\left(\frac{1}{5}\right)$$
   $$= (81 + 9 + 0 + 9 + 81)\left(\frac{1}{5}\right) = 36.0$$
   $$\sigma = \sqrt{36.0} = 6.0$$

   (b) *Solution 2:* $\text{Var}(3X + 5) = (3)^2\text{Var}(X) = (3)^2 4.0 = 6.0$

Example 1.9.1
Example 1.9.2
Example 1.9.3 (Mean may not exist)

Definition 1.9.3. *The* **moment generating function (mgf)** *of a random variable X is $M(t) = E\left(e^{tX}\right)$, for some interval $(-h, h)$ over which the mgf exists.*

Example 1.9.4

Theorem 1.9.2. *The mgf of X uniquely determines the distribution of X, i.e. if the mgf of X is the same as the mgf of a uniform distribution, then X has a uniform distribution.*

Since $M(t) = \int e^{tx} f(x)dx$ then

$$M'(t) = \frac{d}{dt}M(t) = \frac{d}{dt}\int e^{tx} f(x)dx = \int \frac{d}{dt}e^{tx} f(x)dx = \int xe^{tx} f(x)dx$$

Evaluating at $t = 0$,

$$M'(0) = \int xf(x)dx = E(X)$$

Furthermore,

$$M''(t) = \frac{d}{dt}\int xe^{tx} f(x)dx = \int x^2 e^{tx} f(x)dx$$
$$M''(0) = E(X^2)$$

In general, for any positive integer m,

$$M^{(m)}(0) = E(X^m)$$

Example 1.9.5 (mgf may not exist)
Example 1.9.7 (mgf of Normal distribution)