Special Continuous Distributions

Day 6 (1/23/20)

3.3 Gamma and Chi-square

- A continuous random variable X has a **Gamma distribution** with parameters $\alpha > 0$ and $\beta > 0$ if its pdf is
$$f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta},\ 0 < x < \infty \tag{3.3.2}$$
where $\Gamma(\alpha)$ is the **gamma function**
$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$$
If α is a positive integer greater than 1, then $\Gamma(\alpha) = (\alpha - 1)!$.

- Let $w = x/\beta$ so that $dw = \frac{1}{\beta}dx$. Then
$$\int f(x)dx = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx = \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(\frac{x}{\beta}\right)^{\alpha-1} e^{-x/\beta} \left(\frac{1}{\beta}\right) dx$$
$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty w^{\alpha-1} e^{-w} dw = 1$$
The following identity is useful to remembr
$$\int_0^\infty x^{\alpha-1} e^{-x/\beta} dx = \Gamma(\alpha)\beta^\alpha \tag{3.3.2*}$$

- Moment generating function
$$M(t) = E(e^{tX}) = \int_0^\infty e^{tx} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha-1} e^{-x(1-\beta t)/\beta} dx$$
$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \Gamma(\alpha) \left(\frac{\beta}{1-\beta t}\right)^\alpha = \frac{1}{(1-\beta t)^\alpha}$$

- Mean and variance (p.175)
$$E(X) = \alpha\beta$$
$$\text{var}(X) = \alpha\beta^2$$

- $\Gamma(\alpha, \beta)$ in R

  dgamma(x, shape=a, scale=b), pgamma, qgamma, rgamma

  (See Figure 3.3.1 for shapes of density)

Theorem 3.3.1. *Let $X_1, X_2, \ldots, X_n$ be independent $\Gamma(\alpha_i, \beta)$ random variables. Then*
$$\sum_{i=1}^n X_i \textit{ has distribution } \Gamma\left(\sum_{i=1}^n \alpha_i, \beta\right)$$

Proof. Let $Y = \sum_{i=1}^{n} X_i$. Then Y has mgf

$$\begin{aligned}M_Y(t) &= E(e^{tY}) = E(e^{t\sum X_i}) = E(e^{tX_1}e^{tX_2}\cdots e^{tX_n}) = E(e^{tX_1})E(e^{tX_2})\cdots E(e^{tX_n}) \\ &= M_{X_1}(t)\, M_{X_2}(t)\cdots M_{X_n}(t) = \frac{1}{(1-\beta t)^{\alpha_1}}\frac{1}{(1-\beta t)^{\alpha_2}}\cdots\frac{1}{(1-\beta t)^{\alpha_n}} = \frac{1}{(1-\beta t)^{\sum \alpha_i}}\end{aligned}$$

which is the mgf of $\Gamma\left(\sum_{i=1}^{n} \alpha_i, \beta\right)$. □

3.3.1 Special Cases

1. Gamma($\alpha = 1, \beta$) is called Exponential(β)

$$f(x) = \frac{1}{\beta}e^{-x/\beta},\ 0 < x < \infty$$

$$E(X) = \beta$$

$$\text{Var}(X) = \beta^2$$

Remark 3.3.1 (Poisson Processes) Recall that number of accidents in an intersection for a 100-day interval is modeled by a Poisson process with mean $\mu = 2.5$ (or similarly, by a Poisson process with rate of $\lambda = .025$ per day). Let W be the waiting time (in days) until the next event.

$$P(W > 10 \text{ days}) = P(0 \text{ events in 10 days}) = \frac{e^{-.25}(.25)^0}{0!} = e^{-.25}$$

since the expected number of events in 10 days is $\lambda(10) = (.025)(10) = .25$. In general,

$$P(W > t \text{ days}) = P(0 \text{ events in t days}) = \frac{e^{-.025t}(.025t)^0}{0!} = e^{-.025t}$$

$$F_W(t) = P(W \le t) = 1 - P(W > t) = 1 - e^{-.025t}$$

$$f_W(t) = .025e^{-.025t},\ 0 < t < \infty$$

so that waiting time W has distribution Exp($\beta = \frac{1}{.025} = 40$). The expected waiting time is $\beta = 40$ days.

2. Gamma($\alpha = r/2, \beta = 2$) is called χ^2 with r degrees of freedom.

$$f(x) = \frac{1}{\Gamma(r/2)2^{\alpha}}x^{r/2-1}e^{-x/2},\ 0 < x < \infty \tag{3.3.7}$$

$$E(X) = r$$

$$\text{Var}(X) = 2r$$

χ^2 in R: dchisq, pchisq, qchisq, rchisq

Corollary 3.3.1. *Let $X_1, X_2, \ldots, X_n$ be independent χ^2 random variables with r_i degrees of freedom, respectively. Then*

$$\sum_{i=1}^{n} X_i \textit{ has a } \chi^2 \textit{ distribution with } \sum_{i=1}^{n} r_i \textit{ degrees of freedom}$$

Example Using R to confirm Theorem 3.3.1

3.4 Normal Distribution

Definition 3.4.1. *We say that a continuous random variable has a* **normal distribution** *with parameters μ and σ^2 if it has pdf*

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \ -\infty < x < \infty \tag{3.4.6}$$

Properties:

- Notation: $X \sim N(\mu, \sigma^2)$
- Moment generating function: $M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$
- Mean and variance:
$$E(X) = M'(0) = \mu$$
$$\text{Var}(X) = M''(0) - [M'(0)]^2 = \sigma^2$$
- dnorm, pnorm, qnorm, rnorm
- Location-Scale Transformation: If $X \sim N(\mu, \sigma^2)$ then
  1. $aX + b \sim N(a\mu + b, a^2\sigma^2)$
  2. $\frac{X-\mu}{\sigma} \sim N(0, 1)$
- (Empirical Rule) Let $X \sim N(\mu, \sigma^2)$. Then

k	$P(\lvert X-\mu\rvert < k\sigma)$
1	.6827
2	.9545
3	.9973

Theorem 3.4.1. *If $X \sim N(\mu, \sigma^2)$ then $(X-\mu)^2/\sigma^2 \sim \chi^2(1)$*

Theorem 3.4.2. *(Sum of independent normals)*

1. *If $X_1, X_2, \ldots, X_n$ are independent $N(\mu_i, \sigma_i^2)$, then $\sum_{i=1}^n X_i \sim N\left(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2\right)$*

   Proof. $\sum X_i$ has mgf
$$\begin{aligned} M(t) &= E(e^{t\sum X_i}) = E(e^{tX_1}e^{tX_2}\cdots e^{tX_n}) = E(e^{tX_1})E(e^{tX_2})\cdots E(e^{tX_n}) \\ &= (e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2})(e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2})\cdots(e^{\mu_n t + \frac{1}{2}\sigma_n^2 t^2}) \\ &= e^{(\sum \mu_i)t + \frac{1}{2}(\sum \sigma_i^2)t^2} \end{aligned}$$
   which is the mgf of $N\left(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2\right)$. □

2. *If $X_1, X_2, \ldots, X_n$ are independent $N(\mu, \sigma^2)$, then*
$$\overline{X} = (1/n)\sum_{i=1}^n X_i \sim N(\mu, \sigma^2/n)$$
   Proof. By the sum of normals, $\sum_{i=1}^n X_i \sim N(n\mu, n\sigma^2)$. Then by scale transformation, $(1/n)\sum_{i=1}^n X_i \sim N(\mu, \sigma^2/n)$. □