Point Estimation

Day 7 (1/28/20)

4.1.1 Point Estimators

Two sets of assumptions:

- Case 1: $f(x)$ or $p(x)$ is completely unknown.
- Case 2: $f(x)$ or $p(x)$ is known except for the value of some parameters.

Example Maternal Flu Vaccine (handout)

- $p(x) \sim \text{Binomial}(n, p) \quad \leftarrow$ "Birthweight < 2.5 kg"
- $f(x) \sim N(\mu, \sigma^2) \quad \leftarrow$ "Birthweight"
- $f(x) \sim \text{Exp}(\lambda)$

Definition 4.1.1. *Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables. Then $X_1, X_2, \ldots, X_n$ is called a* **random sample** *of size n from the common distribution $f(\cdot)$.*

Definition 4.1.2. *Any function $T(X_1, X_2, \ldots, X_n)$ is called a* **statistic**. *Note that $T(X_1, X_2, \ldots, X_n)$ cannot be a function of unknown parameters.*

Definition 4.1.3. *If $E[T(X_1, X_2, \ldots, X_n)] = \theta$ then T is an* **unbiased estimator** *of θ.*

Example Sample mean is unbiased for population mean

$$\begin{aligned} E(T(X_1, X_2, \ldots, X_n) &= E\left(\frac{X_1 + \cdots + X_n}{n}\right) = E\left(\frac{1}{n}X_1 + \cdots + \frac{1}{n}X_n\right) \\ &= \frac{1}{n}E(X_1) + \cdots + \frac{1}{n}E(X_n) = \frac{1}{n}\mu_1 + \cdots + \frac{1}{n}\mu_n = \mu \end{aligned}$$

5 Maximum likelihood estimation

Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with distribution $f(x; \theta)$. The joint distribution of the sample is

$$f(x_1, \ldots, x_n) = f(x_1; \theta) \cdots f(x_n; \theta) = \prod_{i=1}^{n} f(x_i; \theta)$$

The *likelihood function* is

$$L(\theta) = f(x_1; \theta) \cdots f(x_n; \theta) = \prod_{i=1}^{n} f(x_i; \theta)$$

which looks the same, but is viewed as a function of the parameter θ.
The *maximum likelihood estimate* is the specific value of θ, denoted $\hat{\theta}$, that maximizes $L(\theta)$.

Example 4.1.1 (p.227) Suppose $X \sim \text{Exp}(\theta)$, i.e. $f(x;\theta) = \frac{1}{\theta}e^{-x/\theta}, \quad x \geq 0$. Consider the air conditioner data on p.227 where $X=$ days of operation until service is needed. Given the data below, find the maximum likelihood estimate of θ.

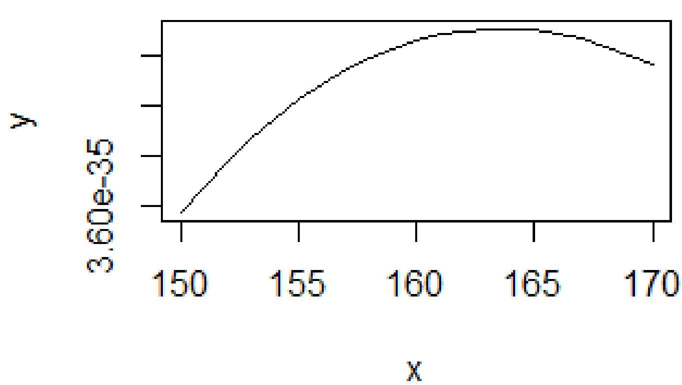
$$359, 413, 25, 130, 90, 50, 50, 487, 102, 194, 55, 74, 97$$

Solution:

```
> ophrs<-c(359,413,25,130,90,50,50,487,102,194,55,74,97)
> (1/10)*exp(-25/10)
[1] 0.0082085
> dexp(25,10)
[1] 2.66919e-108
> dexp(25,1/10)
[1] 0.0082085
> # Vector of densities
> dexp(ophrs,1/10)
 [1] 2.563469e-17 1.157812e-19 8.208500e-03 2.260329e-07 1.234098e-05 6.737947e-04
     6.737947e-04
 [8] 7.077155e-23 3.717032e-06 3.755667e-10 4.086771e-04 6.112528e-05 6.128350e-06
 > prod(dexp(ophrs,1/10))
[1] 4.66652e-106
>
> # Create likelihood function
> Ltheta<-function(theta){prod(dexp(ophrs,1/theta))}
> a<-seq(10,200,by=10)
> a
 [1]  10  20  30  40  50  60  70  80  90 100 110 120 130
[14] 140 150 160 170 180 190 200
> for(i in 1:length(a)){b[i]=Ltheta(a[i])}
> cbind(a,b)
        a            b
 [1,]  10 4.666520e-106
 [2,]  20  8.338859e-64
 [3,]  30  1.048145e-50
 [4,]  40  1.231596e-44
 [5,]  50  2.800197e-41
 [6,]  60  3.129915e-39
 [7,]  70  6.661647e-38
 [8,]  80  5.229430e-37
 [9,]  90  2.166939e-36
[10,] 100  5.846545e-36
[11,] 110  1.169950e-35
[12,] 120  1.889702e-35
```

```
[13,] 130  2.608262e-35
[14,] 140  3.200837e-35
[15,] 150  3.592636e-35
[16,] 160  3.764890e-35
[17,] 170  3.740489e-35
[18,] 180  3.564159e-35
[19,] 190  3.286068e-35
[20,] 200  2.951615e-35

> x<-seq(150,170)
> x
 [1] 150 151 152 153 154 155 156 157 158 159 160 161 162 163 164 165 166 167 168 169 170
> for(i in 1:length(x)){y[i]<-Ltheta(x[i])}
> y
 [1] 3.592636e-35 3.619624e-35 3.644391e-35 3.666951e-35 3.687319e-35 3.705517e-35
     3.721568e-35
 [8] 3.735499e-35 3.747341e-35 3.757127e-35 3.764890e-35 3.770669e-35 3.774504e-35
     3.776437e-35
[15] 3.776509e-35 3.774767e-35 3.771256e-35 3.766024e-35 3.759119e-35 3.750591e-35
     3.740489e-35
>
> plot(x,y,type="l")
```




```
> optimize(Ltheta,interval=c(150,170),maximum=TRUE)
$maximum
[1] 163.5385
> Calculate mean
> mean(ophrs)
[1] 163.5385
```

```
> Check maximum
> prod(dexp(ophrs,rate=1/160))
[1] 3.76489e-35
> prod(dexp(ophrs,rate=1/163))
[1] 3.776437e-35
> prod(dexp(ophrs,rate=1/163.5385))
[1] 3.776704e-35
> prod(dexp(ophrs,rate=1/164))
[1] 3.776509e-35
```

We can analytically, rather than numerically, show that the likelihood function is maximized at the sample mean. The MLE maximizes

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} (\frac{1}{\theta}) e^{-x_i/\theta}$$
$$= (1/\theta)^n e^{-\sum_{i=1}^{n} x_i/\theta}$$

or maximizes $l(\theta) = \ln[L(\theta)] = -n\ln(\theta) - \frac{\sum_{i=1}^{n} X_i}{\theta}$

$$\frac{\partial}{\partial\theta} l(\theta) = \frac{-n}{\theta} + \frac{\sum_{i=1}^{n} X_i}{\theta^2}$$

Equating the RHS to 0, and denoting the solution by $\hat{\theta}$,

$$\frac{-n}{\hat{\theta}} = -\frac{\sum_{i=1}^{n} X_i}{\hat{\theta}^2}$$

Hence

$$\hat{\theta} = \frac{\sum_{i=1}^{n} X_i}{n} = \overline{X}$$

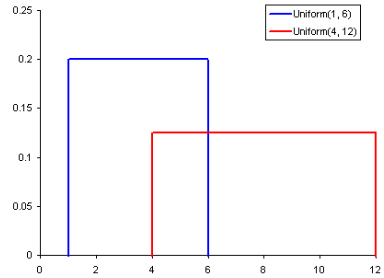
Example 4.1.4 Let $(x_1, x_2, x_3, x_4) = (7.9, 10.5, 4.2, 7.1)$ be a random sample from Unif(0, θ). Find MLE of θ.

Solution

$$L(\theta) = \begin{cases} \left(\frac{1}{\theta}\right)^4, & \text{if } 0 \le x_1 \le \theta,\ 0 \le x_2 \le \theta,\ 0 \le x_3 \le \theta,\ 0 \le x_4 \le \theta, \\ 0, & \text{otherwise} \end{cases}$$
$$= \begin{cases} 0, & \text{if } \theta = 9.0 \\ 0, & \text{if } \theta = 10.0 \\ (1/11)^4, & \text{if } \theta = 11.0 \\ (1/12)^4, & \text{if } \theta = 12.0 \end{cases}$$
$$= \begin{cases} 0, & \text{if } \theta < 10.5 \\ (1/\theta)^4, & \text{if } \theta \ge 10.5 \end{cases}$$

which is maximum at $\hat{\theta} = 10.5$. In general, for Unif(0, θ), the MLE is

$$\hat{\theta}_{\text{MLE}} = \max(x_1, \ldots, x_n)$$



$$f(x_i;\theta) = \begin{cases} \frac{1}{\theta}, & \text{if } 0 \leq x_i \leq \theta \\ 0, & \text{otherwise} \end{cases}$$