Stat 4620: Day 19

(4/2)

Sec. 6.3: Maximum likelihood-based tests

Let $x_1, \ldots, x_n$ be a random sample from a distribution $f(x;\theta)$. Suppose that we want to test

$$H_0 : \theta = \theta_0 \text{ vs } H_1 : \theta \neq \theta_0$$

where θ_0 is a specified value. Let

$$L(\theta) = \prod f(x_i;\theta) \text{ and } l(\theta) = \sum \ln f(x_i;\theta)$$

both of which are maximized by $\hat{\theta}_{mle}$. Define the likelihood ratio test statistic as

$$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta}_{mle})}$$

The **likelihood ratio test** will reject the null hypothesis if

$$\Lambda \leq c$$

where c is chosen so that $P_{H_0}[\Lambda \leq c] = \alpha$.

Example 6.3.1 (Exponential) Let $x_1, \ldots, x_n$ be a random sample from

$$f(x;\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad 0 < x < \infty$$

Conduct a test for $H_0 : \theta = 6.0$ versus $H_1 : \theta \neq 6.0$.

Solution. Recall that $E(X) = \theta$ and $V(X) = \theta^2$. Also,

$$L(\theta) = \prod f(x_i;\theta) = \left(\frac{1}{\theta}\right)^n e^{-\frac{\sum x_i}{\theta}}$$

$$l(\theta) = -n \ln \theta - \frac{\sum x_i}{\theta}$$

$$\frac{\partial}{\partial \theta} l(\hat{\theta}) = -\frac{n}{\theta} + \frac{\sum x_i}{\theta^2}$$

Equating to 0, then

$$\frac{n}{\hat{\theta}} = \frac{\sum x_i}{\hat{\theta}^2} \text{ implies } \hat{\theta} = \frac{\sum x_i}{n} = \overline{x}$$

The LRT statistic is

$$\Lambda = \frac{L(6.0)}{L(\overline{x})} = \frac{\left(\frac{1}{6.0}\right)^n e^{-\frac{\sum x_i}{6.0}}}{\left(\frac{1}{\overline{x}}\right)^n e^{-\frac{\sum x_i}{\overline{x}}}} = \left(\frac{\overline{x}}{6.0}\right)^n e^{-\frac{n\overline{x}}{6.0} + \frac{n\overline{x}}{\overline{x}}} = e^n \left(\frac{\overline{x}}{6.0}\right)^n e^{-n\left(\frac{\overline{x}}{6.0}\right)}$$

$$\equiv g\left(\frac{\overline{x}}{6.0}\right)$$

(Plot here)

Therefore, the LRT rejection region $\Lambda \le c$ is equivalent to

$$\frac{\overline{x}}{6.0} \le a \quad \text{or} \quad \frac{\overline{x}}{6.0} \ge b$$

where a and b depend on c. In practice, instead of calculating a and b from c, we choose a and b such that

$$P_{H_0}\left[\frac{\overline{x}}{6.0} \le a\right] = \frac{\alpha}{2} \text{ and } P_{H_0}\left[\frac{\overline{x}}{6.0} \ge b\right] = \frac{\alpha}{2}$$

so that the probability of false rejection is α. Note that under H_0, the statistic

$$\frac{2\sum x_i}{\theta_0} \sim \chi^2_{2n}$$

i.e. $\frac{2n\overline{x}}{6.0}$ has a χ^2 distribution with $2n$ degrees of freedom. Let $\chi^2_{.025,2n}$ denote the 2.5th percentile. Then

$$.025 = P\left[\frac{2n\overline{x}}{6.0} \le \chi^2_{.025,2n}\right] = P\left[\frac{\overline{x}}{6.0} \le \frac{\chi^2_{.025,2n}}{2n}\right]$$

so that $a = \frac{\chi^2_{.025,2n}}{2n}$. Similarly, $b = \frac{\chi^2_{.975,2n}}{2n}$ because

$$.025 = P\left[\frac{2n\overline{x}}{6.0} \ge \chi^2_{.975,2n}\right] = P\left[\frac{\overline{x}}{6.0} \ge \frac{\chi^2_{.975,2n}}{2n}\right]$$

In summary, the LRT rejects $H_0 : \theta = 6.0$ if

$$\frac{\overline{x}}{6.0} \le \frac{\chi^2_{.025,2n}}{2n} \quad \text{or} \quad \frac{\overline{x}}{6.0} \ge \frac{\chi^2_{.975,2n}}{2n}$$

For illustration purposes, suppose that $n = 10$. Then the LRT for $H_0 : \theta = 6.0$ will reject if

$$\frac{\overline{x}}{6.0} \le \frac{9.591}{20} = .480 \quad \text{or} \quad \frac{\overline{x}}{6.0} \ge \frac{34.170}{20} = 1.708$$

In terms of $\overline{x}$ alone, the rejection rule boundaries are

$$\overline{x} \le 2.88 \quad \text{or} \quad \overline{x} \ge 10.25$$

which seems quite conservative (i.e. observing a sample mean of 3.0 does not allow you to reject $H_0 : \theta = 6.0$). This is an artifact of small sample size. Note that the lower and upper rejection boundaries for $\overline{x}$ are

$$\overline{x} \le \theta_0 \frac{\chi^2_{.025,2n}}{2n} \quad \text{and} \quad \overline{x} \ge \theta_0 \frac{\chi^2_{.975,2n}}{2n} \tag{1}$$

When $\theta_0 = 6.0$, the boundaries in equation (1) are calculated below for increasing sample size n.

n	Lower	Upper
10	2.88	10.25
30	4.05	8.33
100	4.88	7.23
500	5.49	6.54
5000	5.83	6.17

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> # CALCULATIONS IN R
>
> n<-10
> t<-seq(2,12,by=.1)
> y<-exp(n)*(t/6.0)^n*exp(-n*t/6.0)           # Lambda(xbar)
> plot(t,y,type="l")                          # Plot of Lambda(xbar)
> exp(n)*(2.88/6.0)^n*exp(-n*2.88/6.0)        # Lambda(a)
[1] 0.1176911
> exp(n)*(10.25/6.0)^n*exp(-n*10.25/6.0)      # Lambda(b)
[1] 0.1776125
>
> gfn<-function(xarg){exp(n)*(xarg/6.0)^n*exp(-n*xarg/6.0)-.2}   # Find Lambda(?)=.20
> uniroot(gfn,c(0,6))$root                    # Root left of 6.0
[1] 3.206859
> uniroot(gfn,c(6,12))$root                   # Root right of 6.0
[1] 10.07612
>
> pchisq(3.2*2*n/6.0, 2*n)                    # Calculate tail probabilities
[1] 0.04558694
> pchisq(3.2*2*n/6.0, 2*n) + (1-pchisq(10.07*2*n/6.0, 2*n))
[1] 0.07479682
> low<-uniroot(gfn,c(0,6))$root
> hi<-uniroot(gfn,c(6,12))$root
> pchisq(low*2*n/6.0, 2*n) + (1-pchisq(hi*2*n/6.0, 2*n))
[1] 0.0751775
>                                             # Write a function to output tail prob
> lrta<-function(y){
+   gfn<-function(xarg){exp(n)*(xarg/6.0)^n*exp(-n*xarg/6.0)-y}
+   low<-uniroot(gfn,c(0,6))$root
+   hi<-uniroot(gfn,c(6,12))$root
+   pchisq(low*2*n/6.0, 2*n) + (1-pchisq(hi*2*n/6.0, 2*n))
+ }
> lrta(.2)
[1] 0.0751775
> lrta(.15)
[1] 0.05337219
> lrta(.14)
[1] 0.04921543
> lrta(.141)
[1] 0.04962796
> lrta(.142)
[1] 0.0500412
>
> gfn<-function(xarg){exp(n)*(xarg/6.0)^n*exp(-n*xarg/6.0)-.142}
> uniroot(gfn,c(0,6))$root
[1] 2.987776
> uniroot(gfn,c(6,12))$root
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```
[1] 10.56704
> pchisq(2.987776*2*n/6.0, 2*n)
[1] 0.03109521
> 1-pchisq(10.56704*2*n/6.0, 2*n)
[1] 0.0189459
>
> plot(t,y,type="l",ylab="Lambda", xlab="xbar")  # Plot of Lambda(xbar)
> abline(h=.142, col="red")
> axis(2, at=.142, col.axis="red", las=2, cex.axis=0.7, tck=-.01)
> abline(v=c(2.988,10.567), col=c("blue","blue"))
> axis(1, at=c(2.988,10.567), col.axis="blue", las=2, cex.axis=0.7, tck=-.01)
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Example 6.3.2 (Test for the mean of Normal distribution) Let $x_1, \ldots, x_n$ be a random sample from $N(\theta, \sigma^2)$ where σ^2 is known. Conduct a test for $H_0 : \theta = 6.0$ versus $H_1 : \theta \neq 6.0$.

Solution. The likelihood function is

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\theta)^2}{2\sigma^2}} = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{\sum(x_i-\theta)^2}{2\sigma^2}}$$

Recall that $\sum(x_i-\theta)^2 = \sum((x_i-\overline{x})+(\overline{x}-\theta))^2 = \sum(x_i-\overline{x})^2+n(\overline{x}-\theta)^2$, since $\sum(x_i-\overline{x})(\overline{x}-\theta) = 0$.

$$L(\theta) = \left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{\sum(x_i-\overline{x})^2}{2\sigma^2}} e^{-\frac{n(\overline{x}-\theta)^2}{2\sigma^2}}$$

Then

$$\Lambda = \frac{L(6.0)}{L(\overline{x})} = \frac{\left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{\sum(x_i-\overline{x})^2}{2\sigma^2}} e^{-\frac{n(\overline{x}-6.0)^2}{2\sigma^2}}}{\left(\frac{1}{2\pi\sigma^2}\right)^{n/2} e^{-\frac{\sum(x_i-\overline{x})^2}{2\sigma^2}} e^{-\frac{n(\overline{x}-\overline{x})^2}{2\sigma^2}}} = e^{-\frac{n(\overline{x}-6.0)^2}{2\sigma^2}}$$

The rejection region is $\Lambda \leq c$, where c is chosen so that the null probability of rejection equals $\alpha = .05$, i.e.

$$.05 = P\left[\Lambda \leq c\right] = P\left[e^{-\frac{n(\overline{x}-6.0)^2}{2\sigma^2}} \leq c\right] = P\left[\frac{n(\overline{x}-6.0)^2}{\sigma^2} \geq -2\ln c\right] =$$

Now $\frac{\overline{x}-6.0}{\sigma/\sqrt{n}} \sim N(0,1)$ so that $\frac{n(\overline{x}-6.0)^2}{\sigma^2}$ has a χ^2 distribution with 1 degree of freedom. Therefore

$$-2\ln c = \chi^2_{.95,1} \text{ which implies that } c = e^{-\frac{\chi^2_{.95,1}}{2}}$$

In practice, the LRT is written in one of several ways: Reject H_0 if

1. $e^{-\frac{n(\overline{x}-6.0)^2}{2\sigma^2}} \leq e^{-\frac{\chi^2_{.975,1}}{2}}$ [or .1465]
2. $\frac{n(\overline{x}-6.0)^2}{\sigma^2} \geq \chi^2_{.95,1}$ [or 3.84]
3. $\left|\frac{\sqrt{n}(\overline{x}-6.0)}{\sigma}\right| \geq z_{.975}$ [or 1.96]
4. $|\overline{x} - 6.0| \geq z_{.975}\frac{\sigma}{\sqrt{n}}$
5. $\overline{x} \pm z_{.975}\frac{\sigma}{\sqrt{n}}$ does not contain 6.0

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