Stat 4620: Day 20

(4/4)

Sec. 6.3: Asymptotic normality of LRT, Wald-type test, Rao-test

Theorem 10. *Let $x_1, \dots, x_n$ be a random sample from a distribution $f(x;\theta)$. Suppose that we want to test $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$ where θ_0 is a specified value. Assume that (R0)-(R5)hold, and let $\Lambda = \frac{L(\theta_0)}{L(\hat{\theta}_{mle})}$ be the likelihood ratio test statistic. Then*

$$-2 \ln \Lambda \xrightarrow{D} \chi^2_1$$

Comment:

1. Since the LRT rejects for small values of Λ, the theorem says that for large samples, the LRT with size $\alpha = .05$ rejects H_0 if

$$-2 \ln \Lambda \geq \chi^2_{.95,1} = 3.841$$

2. Equivalently, the large sample LRT rejects H_0 if

$$\Lambda \leq e^{-\frac{3.841}{2}} = .1465$$

Example (Exponential, con't.)
Let $x_1, \dots, x_n$ be a random sample from $f(x;\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}}, \quad 0 < x < \infty$. Conduct a test for $H_0 : \theta = 6.0$ versus $H_1 : \theta \neq 6.0$.

Solution. Recall that

$$\Lambda = \frac{L(6.0)}{L(\overline{x})} = e^n \left(\frac{\overline{x}}{6.0} \right)^n e^{-n\left(\frac{\overline{x}}{6.0}\right)}$$

so from Comment (2) above, reject H_0 if

$$e^n \left(\frac{\overline{x}}{6.0} \right)^n e^{-n\left(\frac{\overline{x}}{6.0}\right)} \leq .1465$$

For $n = 10$, this rejection region is equivalent to $\overline{x} \leq 3.006$ or $\overline{x} \geq 10.524$. We have previously calculated two versions of rejection regions for this example, summarized below.

Table 1: Rejection boundaries for Exponential example (n=10)

Test	Approach	Lower	Upper
1	Symmetric tails	2.877	10.251
2	Exact: $\Lambda < .1420$	3.207	10.076
3	Large n: $\Lambda < .1465$	3.006	10.524

■

Proof. Recall that if $l(\theta) = \ln \prod_{i=1}^{n} f(x_i, \theta) = \sum_{i=1}^{n} \ln f(x_i, \theta)$, the mle $\hat{\theta}_n$ satisfies

$$0 = l'(\hat{\theta}_n) = l'(\theta_0) + \left(\hat{\theta}_n - \theta_0\right) l''(\theta_0) + \frac{1}{2}\left(\hat{\theta}_n - \theta_0\right)^2 l'''(\theta_n^*)$$

so that

$$l'(\theta_0) = -\left(\hat{\theta}_n - \theta_0\right) l''(\theta_0) - \frac{1}{2}\left(\hat{\theta}_n - \theta_0\right)^2 l'''(\theta_n^*)$$

Since $-\frac{l''(\theta_0)}{n} \xrightarrow{p} -E[\frac{\partial^2}{\partial\theta^2} \ln f(\theta_0)] = \mathbf{I}(\theta_0)$ and $\sqrt{n}\left(\hat{\theta}_n - \theta_0\right) \xrightarrow{D} N(0, \mathbf{I}^{-1}(\theta_0))$ Then

$$\begin{aligned}
\frac{1}{\sqrt{n}} l'(\theta_0) &= -\sqrt{n}\left(\hat{\theta}_n - \theta_0\right) \frac{l''(\theta_0)}{n} + \frac{1}{2\sqrt{n}}\left(\sqrt{n}(\hat{\theta}_n - \theta_0)\right)^2 \frac{l'''(\theta_n^*)}{n} \\
&= \sqrt{n}\left(\hat{\theta}_n - \theta_0\right) \mathbf{I}(\theta_0) + R_n
\end{aligned}$$

where $R_n \to 0$ as $n \to \infty$. Now

$$\begin{aligned}
-2\ln\Lambda = -2\ln\frac{L(\theta_0)}{L(\hat{\theta}_n)} &= -2\left[\ln L(\theta_0) - \ln L(\hat{\theta}_n)\right] = 2\left[l(\hat{\theta}_n) - l(\theta_0)\right] \\
&= 2\left[\left(\hat{\theta}_n - \theta_0\right) l'(\theta_0) + \frac{(\hat{\theta}_n - \theta_0)^2}{2!} l''(\theta_n^*)\right] \\
&= 2\left[\sqrt{n}\left(\hat{\theta}_n - \theta_0\right) \frac{1}{\sqrt{n}} l'(\theta_0) + \frac{n(\hat{\theta}_n - \theta_0)^2}{2!} \frac{l''(\theta_n^*)}{n}\right] \\
&= 2\left[\sqrt{n}\left(\hat{\theta}_n - \theta_0\right)\left\{\sqrt{n}\left(\hat{\theta}_n - \theta_0\right)\mathbf{I}(\theta_0) + R_n\right\} + \frac{n(\hat{\theta}_n - \theta_0)^2}{2!}\{-\mathbf{I}(\theta_0) + R_n\}\right] \\
&= 2\left[\sqrt{n}\left(\hat{\theta}_n - \theta_0\right)\right]^2 \mathbf{I}(\theta_0) - \left[\sqrt{n}\left(\hat{\theta}_n - \theta_0\right)\right]^2 \mathbf{I}(\theta_0) + R_n \\
&= \left[\sqrt{n}\left(\hat{\theta}_n - \theta_0\right)\right]^2 \mathbf{I}(\theta_0) + R_n \\
&= \left[\frac{\sqrt{n}\left(\hat{\theta}_n - \theta_0\right)}{\sqrt{\frac{1}{\mathbf{I}(\theta_0)}}}\right]^2 + R_n \quad \xrightarrow{D} \quad Z^2 \sim \chi_1^2
\end{aligned}$$

□

Besides the LRT, there are other likelihood-based approaches to testing $H_0 : \theta = \theta_0$

1. Likelihood ratio test: Reject H_0 if

$$-2\ln\Lambda > \chi^2_{\alpha,1}$$

2. Wald-type test: Reject H_0 if

$$W = \frac{n(\hat{\theta} - \theta_0)^2}{\frac{1}{\mathbf{I}(\hat{\theta})}} > \chi^2_{\alpha,1}$$

since $\sqrt{n}\left(\hat{\theta}_n - \theta_0\right) \xrightarrow{D} N(0, \mathbf{I}^{-1}(\theta_0))$ under H_0.

3. Rao scores test: Reject H_0 if

$$S = \frac{(l'(\theta_0))^2}{n\mathbf{I}(\theta_0)} > \chi^2_{\alpha,1}$$

since $\frac{1}{\sqrt{n}} l'(\theta_0) = \sqrt{n}\left(\hat{\theta}_n - \theta_0\right) \mathbf{I}(\theta_0) + R_n \xrightarrow{D} N(0, \mathbf{I}(\theta_0))$ under H_0.

Example (Beta(θ, 1)) Let $x_1, \ldots, x_n$ be a random sample from a distribution with continuous pdf

$$f(x;\theta) = \theta x^{\theta-1},\ 0 < x < 1$$

where $\theta \in (0, \infty)$. We want to test

$$H_0 : \theta = 1 \quad \text{versus} \quad H_1 : \theta \neq 1$$

Solution. First, we find the mle.

$$\ln f = \ln\theta + (\theta - 1)\ln x$$

$$l(\theta) = \sum_{i=1}^{n} \ln f(x_i;\theta) = n\ln\theta + \theta\sum \ln x_i - \sum \ln x_i$$

$$\frac{\partial}{\partial\theta} l(\theta) = \frac{n}{\theta} + \sum \ln x_i$$

Equating to 0 and solving for θ, the mle is

$$\hat{\theta} = -\frac{n}{\sum \ln x_i}$$

Since, $L(\theta) = \theta^n (\prod x_i)^{\theta-1} = \theta^n e^{\ln(\prod x_i)^{\theta-1}} = \theta^n e^{(\theta-1)\sum \ln x_i}$,

$$\begin{aligned} L(\hat{\theta}) &= \left(-\frac{n}{\sum \ln x_i}\right)^n e^{\left(-\frac{n}{\sum \ln x_i} - 1\right)\sum \ln x_i} \\ &= \left(-\sum \ln x_i\right)^{-n} n^n\ e^{-n} e^{-\sum \ln x_i} \end{aligned}$$

Furthermore, $\Lambda(1) = 1$, so the LRT rejects $H_0 : \theta = 1$ if

$$\begin{aligned} -2\ln\Lambda &= -2\left[\ln L(1) - \ln L(\hat{\theta})\right] = -2\left[0 - \ln L(\hat{\theta})\right] \\ &= 2\left[-n\ln\left(-\sum \ln x_i\right) - \sum \ln x_i + n\ln n - n\right] \\ &> \chi^2_{.95,1} = 3.841 \end{aligned}$$

For the Wald-type test, we need $\mathbf{I}(\theta)$.

$$\ln f = \ln\theta + (\theta - 1)\ln x$$

$$\frac{\partial}{\partial\theta}\ln f = \frac{1}{\theta} + \ln x$$

and

$$\mathbf{I}(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\ln f\right] = -E\left[-\frac{1}{\theta^2}\right] = \frac{1}{\theta^2}$$

The Wald-type test rejects $H_0 : \theta = 1$ if

$$W = \frac{n(\hat{\theta} - \theta_0)^2}{\frac{1}{\mathbf{I}(\hat{\theta})}} = \frac{n(\hat{\theta} - 1)^2}{\hat{\theta}^2} = n\left(1 - \frac{1}{\hat{\theta}}\right)^2 = n\left(1 + \frac{\sum \ln x_i}{n}\right)^2 > \chi^2_{.95,1} = 3.841$$

For the Rao scores test, recall that

$$l'(\theta) = \frac{\partial}{\partial\theta} l(\theta) = \frac{n}{\theta} + \sum \ln x_i$$

The scores test rejects $H_0 : \theta = 1$ if

$$S = \frac{[l'(\theta_0)]^2}{n\mathbf{I}(\theta_0)} = \frac{[l'(1)]^2}{n\mathbf{I}(1)} = \frac{(n + \sum \ln x_i)^2}{n} = n\left(1 + \frac{\sum \ln x_i}{n}\right)^2 > \chi^2_{.95,1} = 3.841$$

same as Wald test. ■