Stat 4620: Day 22

(4/11)

Sec. 6.5: Multiparameter testing

Let $x_1, \ldots, x_n$ be a random sample from a distribution with density $f(x; \boldsymbol{\theta})$, where $\boldsymbol{\theta} \in \Omega \subset R^p$, i.e. $\boldsymbol{\theta} = [\theta_1, \ldots, \theta_p]'$. We want to test

$$H_0 : \boldsymbol{\theta} \in \omega \text{ versus } H_1 : \theta \in \omega^c$$

where ω is a proper subset of Ω. The **likelihood ratio test** is generalized to vector parameters as follows: Reject H_0 if

$$\Lambda = \frac{max_{\boldsymbol{\theta} \in \omega} L(\boldsymbol{\theta})}{max_{\boldsymbol{\theta} \in \Omega} L(\boldsymbol{\theta})} = \frac{L(\hat{\boldsymbol{\theta}}_\omega)}{L(\hat{\boldsymbol{\theta}}_\Omega)} \leq c$$

where c is such that

$$max P_{\boldsymbol{\theta} \in \omega}[\Lambda \leq c] = \alpha$$

Example 6.5.1 (Test on mean of normal distribution)
Let $x_1, \ldots, x_n$ be a random sample from $N(\mu, \sigma^2)$. Test

$$H_0 : \mu = \mu_0 \text{ versus } H_1 : \mu \neq \mu_0$$

Solution. Here, we have a two-dimensional parameter vector $\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \mu \\ \sigma^2 \end{bmatrix}$. The null and alternative hypotheses may be written as

$$H_0 : \mu = \mu_0, \sigma^2 \text{ unrestricted} \quad \text{versus } H_1 : \mu \text{ unrestricted}, \sigma^2 \text{ unrestricted}$$

First, we find $max_{\boldsymbol{\theta} \in \Omega} L(\boldsymbol{\theta})$:

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$L(\mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{\sum(x_i-\mu)^2}{2\sigma^2}}$$

$$l(\mu, \sigma^2) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\sigma^2 - \frac{\sum(x_i-\mu)^2}{2\sigma^2}$$

The *normal equations* are

1. $\frac{\partial}{\partial\mu} l(\mu, \sigma^2) = \frac{1}{\sigma^2}\sum(x_i - \mu)^2 = 0$

2. $\frac{\partial}{\partial\sigma^2} l(\mu, \sigma^2) = -\frac{n}{2\sigma^2} + \frac{1}{(\sigma^2)^2}\frac{\sum(x_i-\mu)^2}{2} = 0$

Solving simultaneously for μ and σ^2, we get

$$\hat{\boldsymbol{\theta}}_\Omega = \begin{bmatrix} \hat{\mu} \\ \hat{\sigma^2} \end{bmatrix} = \begin{bmatrix} \overline{x} \\ \frac{\sum(x_i-\overline{x})^2}{n} \end{bmatrix}$$

Plugging into the likelihood,

$$L(\hat{\boldsymbol{\theta}}_\Omega) = L\left(\overline{x}, \frac{\sum(x_i-\overline{x})^2}{n}\right) = \left(\frac{1}{2\pi\frac{\sum(x_i-\overline{x})^2}{n}}\right)^{\frac{n}{2}} e^{-\frac{\sum(x_i-\overline{x})^2}{2\frac{\sum(x_i-\overline{x})^2}{n}}}$$
$$= \left(\frac{1}{2\pi\frac{\sum(x_i-\overline{x})^2}{n}}\right)^{\frac{n}{2}} e^{-\frac{n}{2}}$$

Next, we find $max_{\boldsymbol{\theta}\in\omega}L(\boldsymbol{\theta})$:

$$max_{\boldsymbol{\theta}\in\omega}L(\boldsymbol{\theta}) = max_{\sigma^2>0}L(\mu_0,\sigma^2)$$
$$= max_{\sigma^2>0}L(\mu_0,\sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{\sum(x_i-\mu_0)^2}{2\sigma^2}}$$

$$\ln L(\mu_0,\sigma^2) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{\sum(x_i-\mu_0)^2}{2\sigma^2}$$
$$\frac{\partial}{\partial\sigma^2}\ln L(\mu_0,\sigma^2) = -\frac{n}{2\sigma^2} + \frac{\sum(x_i-\mu_0)^2}{2(\sigma^2)^2}$$

Equating to 0,

$$\frac{n}{2\hat{\sigma}^2} = \frac{\sum(x_i-\mu_0)^2}{2(\hat{\sigma}^2)^2} \quad\Rightarrow\quad \hat{\sigma}^2 = \frac{\sum(x_i-\mu_0)^2}{n} =$$

Plugging into the likelihood,

$$L(\hat{\boldsymbol{\theta}}_\omega) = L\left(\mu_0, \frac{\sum(x_i-\mu_0)^2}{n}\right)\left(\frac{1}{2\pi\frac{\sum(x_i-\mu_0)^2}{n}}\right)^{\frac{n}{2}} e^{-\frac{\sum(x_i-\mu_0)^2}{2\frac{\sum(x_i-\mu_0)^2}{n}}}$$
$$= \left(\frac{1}{2\pi\frac{\sum(x_i-\mu_0)^2}{n}}\right)^{\frac{n}{2}} e^{-\frac{n}{2}}$$

The test statistic is

$$\Lambda = \frac{L(\hat{\boldsymbol{\theta}}_\omega)}{L(\hat{\boldsymbol{\theta}}_\Omega)} = \frac{\left(\frac{1}{2\pi\frac{\sum(x_i-\mu_0)^2}{n}}\right)^{\frac{n}{2}} e^{-\frac{n}{2}}}{\left(\frac{1}{2\pi\frac{\sum(x_i-\overline{x})^2}{n}}\right)^{\frac{n}{2}} e^{-\frac{n}{2}}} = \left(\frac{\sum(x_i-\overline{x})^2}{\sum(x_i-\mu_0)^2}\right)^{\frac{n}{2}}$$
$$= \left(\frac{\sum(x_i-\overline{x})^2}{\sum(x_i-\overline{x})^2 + n(\overline{x}-\mu_0)^2}\right)^{\frac{n}{2}} = \left(\frac{\sum(x_i-\overline{x})^2 + n(\overline{x}-\mu_0)^2}{\sum(x_i-\overline{x})^2}\right)^{-\frac{n}{2}}$$
$$= \left(1 + \frac{n(\overline{x}-\mu_0)^2}{\sum(x_i-\overline{x})^2}\right)^{-\frac{n}{2}} \leq c$$

if and only if

$$\frac{n(\overline{x}-\mu_0)^2}{\sum(x_i-\overline{x})^2} \geq c^{-\frac{2}{n}} - 1$$

if and only if

$$\left|\frac{n(\overline{x}-\mu_0)^2}{S}\right| \geq \sqrt{(n-1)\left(c^{-\frac{2}{n}}-1\right)}$$

The random variable on the left side has a t distribution with $n-1$ degrees of freedom if H_0 is true. If we let the right side equal $t_{.975,n-1}$, then

$$P(\Lambda \leq c) = P\left(\left|\frac{n(\overline{x}-\mu_0)^2}{S}\right| \geq t_{.975,n-1}\right) = .05$$

So the likelihood ratio test rejects $H_0 : \mu = \mu_0$ if

$$\left|\frac{n(\overline{x}-\mu_0)^2}{S}\right| \geq t_{.975,n-1}$$

or equivalently,

$$\Lambda \leq \left(\frac{t^2_{.975,n-1}}{n-1}+1\right)^{-\frac{n}{2}}$$

■

Example 6.5.3 (Equality of proportions)
Let $\mathbf{x} = (x_1, \ldots, x_{n_1})$ be a random sample of size n_1 from Bernoulli(p_1), i.e. the $\{x_i\}$ are independent with probability mass function

x_i	1	0
$p(x_i)$	p_1	$1-p_1$

or $p(x_i) = p_1^{x_i}(1-p_1)^{1-x_i},\ x_i = 0, 1$. Similarly, let $\mathbf{y} = (y_1, \ldots, y_{n_2})$ be a random sample of size n_2 from Bernoulli(p_2). The hypotheses of interest are

$$H_0 : p_1 = p_2 \ \text{ versus } \ H_1 : p_1 \neq p_2$$

The combined sample has likelihood

$$\begin{aligned} L(\boldsymbol{\theta}) = L(p_1, p_2; \mathbf{x}, \mathbf{y}) &= \prod_{i=1}^{n_1} p(x_i) \prod_{j=1}^{n_2} p(y_j) \\ &= p_1^{\sum x_i}(1-p_1)^{n_1-\sum x_i}\, p_2^{\sum y_j}(1-p_2)^{n_2-\sum y_j} \end{aligned}$$

$$\ln L(\boldsymbol{\theta}) = \sum x_i \ln p_1 + \left(n_1 - \sum x_i\right)\ln(1-p_1) + \sum y_j \ln p_2 + \left(n_2 - \sum y_j\right)\ln(1-p_2)$$

The normal equations are

1. $\frac{\partial}{\partial p_1}\ln L(\boldsymbol{\theta}) = \frac{\sum x_i}{p_1} - \frac{n_1 - \sum x_i}{1-p_1}$
2. $\frac{\partial}{\partial p_2}\ln L(\boldsymbol{\theta}) = \frac{\sum y_j}{p_2} - \frac{n_2 - \sum y_j}{1-p_2}$

Equating (1) to 0,

$$\frac{\sum x_i}{\hat{p}_1} = \frac{n_1 - \sum x_i}{1-\hat{p}_1} \text{ implies } (1-\hat{p}_1)\sum x_i = \hat{p}_1\left(n_1 - \sum x_i\right)$$

or $\hat{p}_1 = \frac{\sum x_i}{n_1} = \overline{x}$. Similarly, equating (2) to 0 gives $\hat{p}_2 = \frac{\sum y_j}{n_2} = \overline{y}$. Therefore

$$\hat{\boldsymbol{\theta}}_\Omega = \begin{pmatrix} \hat{p}_1 \\ \hat{p}_2 \end{pmatrix} = \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix}$$

Then,

$$\begin{aligned} L(\hat{\boldsymbol{\theta}}_\Omega) = L(\hat{p}_1, \hat{p}_2) &= \hat{p}_1^{\sum x_i}(1-\hat{p}_1)^{n_1-\sum x_i}\ \hat{p}_2^{\sum y_j}(1-\hat{p}_2)^{n_2-\sum y_j} \\ &= \left(\frac{\hat{p}_1}{1-\hat{p}_1}\right)^{\sum x_i}(1-\hat{p}_1)^{n_1}\left(\frac{\hat{p}_2}{1-\hat{p}_2}\right)^{\sum y_j}(1-\hat{p}_2)^{n_2} \\ &= \left(\frac{\hat{p}_1}{1-\hat{p}_1}\right)^{n_1\hat{p}_1}(1-\hat{p}_1)^{n_1}\left(\frac{\hat{p}_2}{1-\hat{p}_2}\right)^{n_2\hat{p}_2}(1-\hat{p}_2)^{n_2} \end{aligned}$$

To find $\hat{\boldsymbol{\theta}}_\omega$, note that under H_0, we have $p_1 = p_2 = p$, so that we are maximizing the likelihood

$$\begin{aligned} L(\boldsymbol{\theta}) &= p^{\sum x_i}(1-p)^{n_1-\sum x_i}\ p^{\sum y_j}(1-p)^{n_2-\sum y_j} \\ &= p^{\sum x_i+\sum y_j}\ (1-p)^{n_1+n_2-\sum x_i-\sum y_j} \end{aligned}$$

Taking derivative with respect to p and equating to 0, we get

$$\hat{p} = \frac{\sum x_i + \sum y_j}{n_1+n_2}$$

Note that $\sum x_i + \sum y_j = (n_1+n_2)\hat{p}$. Now

$$\begin{aligned} L(\hat{\boldsymbol{\theta}}_\omega) &= \hat{p}^{\sum x_i+\sum y_j}\ (1-\hat{p})^{n_1+n_2-\sum x_i-\sum y_j} \\ &= \left(\frac{\hat{p}}{1-\hat{p}}\right)^{\sum x_i+\sum y_j}(1-\hat{p})^{n_1+n_2} \\ &= \left(\frac{\hat{p}}{1-\hat{p}}\right)^{(n_1+n_2)\hat{p}}(1-\hat{p})^{n_1+n_2} \end{aligned}$$

The likelihood ratio test is

$$\begin{aligned} \Lambda = \frac{L(\hat{\boldsymbol{\theta}}_\omega)}{L(\hat{\boldsymbol{\theta}}_\Omega)} &= \frac{\left(\frac{\hat{p}}{1-\hat{p}}\right)^{(n_1+n_2)\hat{p}}(1-\hat{p})^{n_1+n_2}}{\left(\frac{\hat{p}_1}{1-\hat{p}_1}\right)^{n_1\hat{p}_1}(1-\hat{p}_1)^{n_1}\left(\frac{\hat{p}_2}{1-\hat{p}_2}\right)^{n_2\hat{p}_2}(1-\hat{p}_2)^{n_2}} \\ &= \left(\frac{\frac{\hat{p}}{1-\hat{p}}}{\frac{\hat{p}_1}{1-\hat{p}_1}}\right)^{n_1\hat{p}_1}\left(\frac{\frac{\hat{p}}{1-\hat{p}}}{\frac{\hat{p}_2}{1-\hat{p}_2}}\right)^{n_2\hat{p}_2}\left(\frac{1-\hat{p}}{1-\hat{p}_1}\right)^{n_1}\left(\frac{1-\hat{p}}{1-\hat{p}_2}\right)^{n_2} \leq c \end{aligned}$$

Theorem 6.5.1. *Let $x_1, \ldots, x_n$ be a random sample from a distribution $f(x;\boldsymbol{\theta})$ for $\boldsymbol{\theta} \in \Omega$ with dimension p. Let ω be a proper subset of Ω with dimension $p-q$. Assume the regularity conditions hold, including existence of unique mle solutions $\hat{\boldsymbol{\theta}}_\Omega$ and $\hat{\boldsymbol{\theta}}_\omega$. Under $H_0 : \boldsymbol{\theta} \in \omega$,*

$$-2\ln\Lambda \xrightarrow{D} \chi^2_q$$

Comments:

1. The degrees of freedom q is the difference between the dimensions of Ω and ω.

2. Another interpretation: q is the number of constraints represented by the null hypothesis, hence the term *degrees of freedom*.

3. Since the LRT rejects for small values of Λ (or large values of $-2\ln\Lambda$), Theorem 6.5.1 says that the LRT with (approximate) size α is

$$-2\ln\Lambda \geq \chi^2_{1-\alpha,q}$$

or equivalently, $\Lambda \leq e^{-\frac{\chi^2_{1-\alpha,q}}{2}}$

Example 6.5.3 con't The full model parameter space (p_1, p_2) has dimension $p = 2$. The reduced H_0 parameter space p has dimesion $p - q = 1$. The difference is $q = 1$. This is also the number of constraints imposed by $H_0 : p_1 = p_2$. So the likelihood ratio test with size $\alpha = .05$ is

$$-2\ln\Lambda \geq \chi^2_{.95,1df} = 3.841$$

or equivalently

$$\left(\frac{\frac{\hat{p}}{1-\hat{p}}}{\frac{\hat{p}_1}{1-\hat{p}_1}}\right)^{n_1\hat{p}_1}\left(\frac{\frac{\hat{p}}{1-\hat{p}}}{\frac{\hat{p}_2}{1-\hat{p}_2}}\right)^{n_2\hat{p}_2}\left(\frac{1-\hat{p}}{1-\hat{p}_1}\right)^{n_1}\left(\frac{1-\hat{p}}{1-\hat{p}_2}\right)^{n_2} \leq .1465$$

Suppose that

$$n_1 = 50, n_2 = 50, \hat{p}_1 = .30, \hat{p}_2 = .34$$

will the LRT reject $H_0 : p_1 = p_2$?

```
    > n1<-50
> n2<-50
> x1<-15
> x2<-17
> p1hat<-x1/n1
> p2hat<-x2/n2
> phat<-(n1*p1hat+n2*p2hat)/(n1+n2)
> phat
[1] 0.32
> Lambda<-((phat*(1-p1hat))/(p1hat*(1-phat)))^(n1*p1hat)*((phat*(1-p2hat))/(p2hat*(1-phat)))^(
> Lambda
[1] 0.9121446
```

So the test does not reject. How far do the two proportions need to be for the test to reject?

```
> p2hat<-seq(.32,.50,by=.01)
> Lambda<-((phat*(1-p1hat))/(p1hat*(1-phat)))^(n1*p1hat)*((phat*(1-p2hat))/(p2hat*(1-phat)))^(
> cbind(p2hat,Lambda)
      p2hat     Lambda
 [1,]  0.32 0.95457795
 [2,]  0.33 0.94373208
 [3,]  0.34 0.91214459
 [4,]  0.35 0.86218570
 [5,]  0.36 0.79724542
 [6,]  0.37 0.72136912
 [7,]  0.38 0.63886128
 [8,]  0.39 0.55390847
```

```
 [9,]  0.40 0.47026321
[10,]  0.41 0.39101685
[11,]  0.42 0.31847291
[12,]  0.43 0.25411761
[13,]  0.44 0.19867208
[14,]  0.45 0.15220380
[15,]  0.46 0.11427202
[16,]  0.47 0.08408371
[17,]  0.48 0.06064095
[18,]  0.49 0.04286666
[19,]  0.50 0.02970185
> n1<-200
> n2<-200
> Lambda2<-((phat*(1-p1hat))/(p1hat*(1-phat)))^(n1*p1hat)*((phat*(1-p2hat))/(p2hat*(1-phat)))^
> n1<-2000
> n2<-2000
> Lambda3<-((phat*(1-p1hat))/(p1hat*(1-phat)))^(n1*p1hat)*((phat*(1-p2hat))/(p2hat*(1-phat)))^
> cbind(p2hat,Lambda,Lambda<.1465,Lambda2,Lambda2<.1465, Lambda3, Lambda3<.1465)
      p2hat     Lambda        Lambda2        Lambda3
 [1,]  0.32 0.95457795 0 8.303202e-01 0 1.557600e-01 0
 [2,]  0.33 0.94373208 0 7.932222e-01 0 9.861630e-02 1
 [3,]  0.34 0.91214459 0 6.922369e-01 0 2.526664e-02 1
 [4,]  0.35 0.86218570 0 5.525903e-01 0 2.654805e-03 1
 [5,]  0.36 0.79724542 0 4.039877e-01 0 1.157927e-04 1
 [6,]  0.37 0.72136912 0 2.707885e-01 0 2.119834e-06 1
 [7,]  0.38 0.63886128 0 1.665813e-01 0 1.645367e-08 1
 [8,]  0.39 0.55390847 0 9.413520e-02 1 5.464121e-11 1
 [9,]  0.40 0.47026321 0 4.890621e-02 1 7.827811e-14 1
[10,]  0.41 0.39101685 0 2.337663e-02 1 4.873252e-17 1
[11,]  0.42 0.31847291 0 1.028703e-02 1 1.327089e-20 1
[12,]  0.43 0.25411761 0 4.170029e-03 1 1.589976e-24 1
[13,]  0.44 0.19867208 0 1.557928e-03 1 8.423130e-29 1
[14,]  0.45 0.15220380 0 5.366634e-04 1 1.981616e-33 1
[15,]  0.46 0.11427202 1 1.705138e-04 1 2.077763e-38 1
[16,]  0.47 0.08408371 1 4.998590e-05 1 9.738112e-44 1
[17,]  0.48 0.06064095 1 1.352272e-05 1 2.044745e-49 1
[18,]  0.49 0.04286666 1 3.376593e-06 1 1.926582e-55 1
[19,]  0.50 0.02970185 1 7.782763e-07 1 0.000000e+00 1
```