Section 4.2

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1 Confidence Intervals

From luck-of-the-draw, or chance variability, any point estimate $\hat{\theta}$ is likely to miss θ. By how much?

Definition 4.2.1

Let $X_1, X_2, \ldots, X_n$ be a random sample from $f(x;\theta)$. Let $0 < \alpha < 1$ be a prespecified value, usually .05. Let $L(x_1, \ldots, x_n)$ and $U(x_1, \ldots, x_n)$ be statistics. We say that (L, U) is a $(1-\alpha)100\%$ confidence interval for θ if

$$P[L \leq \theta \leq U] = 1 - \alpha$$

$1 - \alpha$ is called the *confidence coefficient*, usually .95.

Examples

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\theta, \sigma^2)$, with σ^2 known. Then

$$P\left[\overline{X} - 1.96\frac{\sigma}{\sqrt{n}} \leq \theta \leq \overline{X} + 1.96\frac{\sigma}{\sqrt{n}}\right] = .95$$

2. Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n = 25$ from a symmetric continuous density $f(\cdot)$ with median θ. Let $X_{(1)}, X_{(2)}, \ldots, X_{(25)}$ denote the *ordered* sample. Then

$$P[X_{(8)} \leq \theta \leq X_{(18)}] \doteq .95$$

Proof. The event $\left[\theta < X_{(8)}\right]$ occurs if and only if *7 or fewer observations are less than* θ. This happens with probability $P[B \leq 7] = .021$ where $B \sim$Binomial$(n = 25, p = .50)$. Similarly, $\left[X_{(18)} < \theta\right]$ occurs with .021 probability. Then

$$P[X_{(8)} \leq \theta \leq X_{(18)}] = 1 - P\left[\theta < X_{(8)}\right] - P\left[X_{(18)} < \theta\right] = 1 - .021 - .021 = 0.958$$

□

1.1 A pivot method for constructing confidence intervals

Let $X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)$ where σ^2 is known, so that there is only one parameter μ to estimate. It can be shown that $\overline{X} \sim N(\mu, \frac{\sigma^2}{n})$, or similarly

$$\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

Then

$$
\begin{aligned}
.95 &= P\left[-1.96 \le \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \le 1.96\right] \\
&= P\left[-1.96\frac{\sigma}{\sqrt{n}} \le \overline{X}-\mu \le 1.96\frac{\sigma}{\sqrt{n}}\right] \\
&= P\left[-\overline{X}-1.96\frac{\sigma}{\sqrt{n}} \le -\mu \le -\overline{X}+1.96\frac{\sigma}{\sqrt{n}}\right] \\
&= P\left[\overline{X}-1.96\frac{\sigma}{\sqrt{n}} \le \mu \le \overline{X}+1.96\frac{\sigma}{\sqrt{n}}\right] \\
&\equiv P[L(X_1,\dots X_n) \le \mu \le U(X_1,\dots X_n)]
\end{aligned}
$$

What if σ^2 is unknown? Let $S^2 = (1/(n-1)\sum(X_i-\mu)^2$. Student (1908) showed that

$$\frac{\overline{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$$

or the t distribution with $n-1$ degrees of freedom. Let $t_{.025,n-1}$ denote the 97.5th percentile (or upper 2.5th percentile) of this distribution. Then

$$
\begin{aligned}
.95 &= P\left[-t_{.025,n-1} \le \frac{\overline{X}-\mu}{S/\sqrt{n}} \le t_{.025,n-1}\right] \\
&\vdots \\
&\vdots \\
&= P\left[\overline{X}-t_{.025,n-1}\frac{S}{\sqrt{n}} \le \mu \le \overline{X}+t_{.025,n-1}\frac{S}{\sqrt{n}}\right] \\
&\equiv P[L(X_1,\dots X_n) \le \mu \le U(X_1,\dots X_n)]
\end{aligned}
$$

In general,

$$1-\alpha = P\left[\overline{X}-t_{\alpha/2,n-1}\frac{S}{\sqrt{n}} \le \mu \le \overline{X}+t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}\right]$$

Comments:

1. For small n, the replacement of σ by S in the denominator adds substantial variability. For large n the effect of substitution gets progressively muted, so that the t_{n-1} distribution is close to $N(0,1)$. (See t-table)

2. $S/\sqrt{n}$ is called the *standard error of the mean*, or SE of the mean. It represents the expected error of estimation.

$$
\begin{aligned}
E|\overline{X}-\mu| &\doteq \sqrt{E(\overline{X}-\mu)^2} = \sqrt{\mathrm{Var}(\overline{X})} = \sqrt{\sigma^2/n} \\
&= \sigma/\sqrt{n} \\
&\doteq S/\sqrt{n}
\end{aligned}
$$

3. Since SE=$S/\sqrt{n}$ is an estimate of standard deviation of $\overline{X}$, then $P[|\overline{X}-\mu| \leq \text{SE}] \doteq .68$. In general

$$P[|\overline{X}-\mu| \leq t_{\alpha/2,n-1}\text{SE}] \doteq 1-\alpha$$

so $t_{\alpha/2,n-1}$SE is called a $(1-\alpha)100\%$ *margin of error* for estimating the mean μ. In particular, a 95% margin of error is

$$t_{.025,n-1}\frac{S}{\sqrt{n}} \doteq 2\frac{S}{\sqrt{n}}$$

if n is reasonable large, say, $n \geq 30$.

Theorem 4.2.1: Central Limit Theorem

Let $X_1, \ldots, X_n$ be a random sample from a population with distribution $f(\cdot)$ that has mean μ and variance σ^2. Then the distribution of

$$\frac{\overline{X}-\mu}{S/\sqrt{n}}$$

converges to $N(0,1)$ as n approaches infinity.

Comment: The most important point of the theorem is that convergence to standard nrmal occurs *regardless of the shape of the underlying distribution.*

Example 4.2.2

If n is large, then

$$\begin{aligned} 1-\alpha &= P\left[-z_{\alpha/2} \leq \frac{\overline{X}-\mu}{S/\sqrt{n}} \leq z_{\alpha/2}\right] \\ &\vdots \\ &\vdots \\ &= P\left[\overline{X}+z_{\alpha/2}\frac{S}{\sqrt{n}} \leq \mu \leq \overline{X}+z_{\alpha/2}\frac{S}{\sqrt{n}}\right] \end{aligned}$$

so that $(\overline{X}-z_{\alpha/2}\frac{S}{\sqrt{n}}, \overline{X}+z_{\alpha/2}\frac{S}{\sqrt{n}})$ is a large sample confidence interval for μ.

Calculating t-percentiles in R

```
> df<-c(seq(1,40),50,100,1000,10000)
> t.10<-qt(.90,df)
> t.05<-qt(.95,df)
> t.025<-qt(.975,df)
> t.005<-qt(.995,df)
> cbind(df,t.10,t.05,t.025,t.005)
```

	df	t.10	t.05	t.025	t.005
[1,]	1	3.077684	6.313752	12.706205	63.656741
[2,]	2	1.885618	2.919986	4.302653	9.924843
[3,]	3	1.637744	2.353363	3.182446	5.840909
[4,]	4	1.533206	2.131847	2.776445	4.604095
[5,]	5	1.475884	2.015048	2.570582	4.032143
[6,]	6	1.439756	1.943180	2.446912	3.707428
[7,]	7	1.414924	1.894579	2.364624	3.499483
[8,]	8	1.396815	1.859548	2.306004	3.355387
[9,]	9	1.383029	1.833113	2.262157	3.249836
[10,]	10	1.372184	1.812461	2.228139	3.169273
[11,]	11	1.363430	1.795885	2.200985	3.105807
[12,]	12	1.356217	1.782288	2.178813	3.054540
[13,]	13	1.350171	1.770933	2.160369	3.012276
[14,]	14	1.345030	1.761310	2.144787	2.976843
[15,]	15	1.340606	1.753050	2.131450	2.946713
[16,]	16	1.336757	1.745884	2.119905	2.920782
[17,]	17	1.333379	1.739607	2.109816	2.898231
[18,]	18	1.330391	1.734064	2.100922	2.878440
[19,]	19	1.327728	1.729133	2.093024	2.860935
[20,]	20	1.325341	1.724718	2.085963	2.845340
[21,]	21	1.323188	1.720743	2.079614	2.831360
[22,]	22	1.321237	1.717144	2.073873	2.818756
[23,]	23	1.319460	1.713872	2.068658	2.807336
[24,]	24	1.317836	1.710882	2.063899	2.796940
[25,]	25	1.316345	1.708141	2.059539	2.787436
[26,]	26	1.314972	1.705618	2.055529	2.778715
[27,]	27	1.313703	1.703288	2.051831	2.770683
[28,]	28	1.312527	1.701131	2.048407	2.763262
[29,]	29	1.311434	1.699127	2.045230	2.756386
[30,]	30	1.310415	1.697261	2.042272	2.749996
[31,]	31	1.309464	1.695519	2.039513	2.744042
[32,]	32	1.308573	1.693889	2.036933	2.738481
[33,]	33	1.307737	1.692360	2.034515	2.733277
[34,]	34	1.306952	1.690924	2.032245	2.728394
[35,]	35	1.306212	1.689572	2.030108	2.723806
[36,]	36	1.305514	1.688298	2.028094	2.719485
[37,]	37	1.304854	1.687094	2.026192	2.715409
[38,]	38	1.304230	1.685954	2.024394	2.711558
[39,]	39	1.303639	1.684875	2.022691	2.707913
[40,]	40	1.303077	1.683851	2.021075	2.704459
[41,]	50	1.298714	1.675905	2.008559	2.677793
[42,]	100	1.290075	1.660234	1.983972	2.625891
[43,]	1000	1.282399	1.646379	1.962339	2.580755
[44,]	10000	1.281636	1.645006	1.960201	2.576321