Moment Generating Functions

1 Univariate Case

1. Definition The moment generating function (m.g.f.) of a random variable \(X\), denoted \(M_X(t)\) or simply \(M(t)\) if there’s no confusion, is defined as

\[
M(t) = E e^{tX}
\]

when \(E e^{tX}\) exists for \(t\) over a neighborhood of 0.

2. Selected Properties

(a) \(M(0) = 1\).
(b) \(M_{aX+b}(t) = e^{bt}M_X(at)\) for constants \(a\) and \(b\).
(c) The moments \(E X^n = M^{(n)}(0), n = 1, 2, \ldots\).
(d) The m.g.f. uniquely defines a distribution.

3. MGF of Normal Variate If \(X \sim N(\mu, \sigma^2)\) then \(M_X(t) = \exp\{\mu t + \frac{1}{2} \sigma^2 t^2\}\).

Proof Note that \(X = \mu + \sigma Z\) where \(Z \sim N(0, 1)\). So for \(t \in \mathbb{R}\),

\[
M_X(t) = M_{\mu+\sigma Z}(t) = e^{\mu t}M_Z(\sigma t).
\]

Now for \(t \in \mathbb{R}\),

\[
M_Z(t) = \int_{\mathbb{R}} e^{tz} \phi(z) dz = \exp\left\{ \frac{t^2}{2} \right\} \int_{\mathbb{R}} f(z) df = \exp\left\{ \frac{t^2}{2} \right\}
\]

where \(\phi(\cdot)\) and \(f(\cdot)\) are the p.d.f.’s of \(N(0, 1)\) and \(N(t, 1)\), respectively. The result follows.

4. MGF of Chi-square Variate If \(X \sim \chi^2(r)\) then \(M_X(t) = (1 - 2t)^{-r/2}\) for \(t < \frac{1}{2}\).

Proof Denote \(f(\cdot)\) and \(g(\cdot)\), respectively, the p.d.f.’s of \(\chi^2(r) \equiv \text{gamma}(\frac{r}{2}, 2)\) and \(\text{gamma}(\frac{r}{2}, (\frac{1}{2} - t)^{-1})\), provided that \(t < \frac{1}{2}\). Now for \(t < \frac{1}{2}\),

\[
M_X(t) = \int_0^\infty e^{tx} f(x) dx = 2^{-\frac{r}{2}} (\frac{1}{2} - t)^{-\frac{r}{2}} \int_0^\infty g(x) dx = 2^{-\frac{r}{2}} (\frac{1}{2} - t)^{-\frac{r}{2}} = (1 - 2t)^{-r/2}.
\]
5. MGF of Noncentral Chi-square Variate

If \( X \sim \chi^2(r, \lambda) \), \( \lambda > 0 \), then

\[
M_X(t) = (1 - 2t)^{-\frac{r}{2}} \exp \left\{ \frac{\lambda t}{1 - 2t} \right\}.
\]

**Proof**

For \( t < \frac{1}{2} \),

\[
M_X(t) = \sum_{i=0}^{\infty} \frac{(\frac{\lambda}{2})^i}{i!} M_{\chi^2(r+2i)}(t) = (1 - 2t)^{-\frac{r}{2}} \exp \left\{ \frac{\lambda t}{2(1 - 2t)} - \frac{\lambda}{2} \right\} \sum_{i=0}^{\infty} p(i) = (1 - 2t)^{-\frac{r}{2}} \exp \left\{ \frac{\lambda t}{1 - 2t} \right\}.
\]

The function \( p(\cdot) \) above is the p.m.f. of a Poisson distribution with parameter (mean occurrence rate)

\[
\frac{\lambda}{2(1 - 2t)}.
\]

6. Quadratic Froms Using MGFs (item 1.(e) in § A.2)

\[
(1 - 2t)^{-\frac{r}{2}} = M_Q(t) = M_{Q_1+Q_2}(t) = E e^{Q_1t} e^{Q_2t} = E e^{Q_1t} \times E e^{Q_2t} \quad \text{(by the fact stated in item 2 in § A.4.1)}
\]

\[
M_{Q_1}(t)M_{Q_2}(t) = (1 - 2t)^{-\frac{r_1}{2}} M_{Q_2}(t).
\]

It follows then that for \( t < \frac{1}{2} \),

\[
M_{Q_2}(t) = (1 - 2t)^{-\frac{r_1-r_2}{2}} = (1 - 2t)^{-\frac{r_2}{2}}.
\]

Thus, \( Q_2 \sim \chi^2(r_2) \).

Note that the proof for noncentral chi-square case (item 2.(f) in § A.2) is similar.

7. Finite Sum Using MGFs (item 2.(e) in § A.2)

\[
M_S(t) = E \prod_{i=1}^{k} e^{tX_i} = \prod_{i=1}^{k} E e^{tX_i} \quad \text{(independence)}
\]

\[
= \prod_{i=1}^{k} M_{X_i}(t) = \prod_{i=1}^{k} (1 - 2t)^{-\frac{r_i}{2}} \exp \left\{ \frac{\lambda_i t}{1 - 2t} \right\}
\]

\[
= (1 - 2t)^{-\sum_{i=1}^{k} \frac{r_i}{2}} \exp \left\{ \sum_{i=1}^{k} \frac{\lambda_i t}{1 - 2t} \right\} = (1 - 2t)^{-\frac{r}{2}} \exp \left\{ \frac{\lambda t}{1 - 2t} \right\}.
\]

Consequently, \( S \sim \chi^2(r, \lambda) \).
2 Multivariate Case

Unless otherwise stated, all vectors (random or constant) are of $p \times 1$.

1. **Definition** The m.g.f. of a random vector $X$, denoted $M_X(t)$ or $M(t)$, is defined as

$$M(t) = E e^{t'X}$$

if the expected value on the right hand side of the above exists for $t$ over a neighborhood of $0$.

2. **Selected Properties**

   (a) $M(0) = 1$.

   (b) $$\left. \frac{dM(t)}{dt} \right|_{t=0} = \mu.$$

   Note that for a real-valued function $h(\cdot)$,

   $$\frac{dh(t)}{dt} = \begin{bmatrix} \frac{\partial h(t)}{\partial t_1} \\ \vdots \\ \frac{\partial h(t)}{\partial t_p} \end{bmatrix}.$$

   (c) $$\left. \frac{d^2M(t)}{dt^2} \right|_{t=0} = [E X_i X_j]_{p \times p}, \text{ the matrix of product moments.}$$

   (d) For a $q \times p$ constant matrix $A$ and a $q \times 1$ constant vector $b$,

   $$M_{AX+b}(t) = e^{b't}M_X(A't)$$

   where $t$ is $q \times 1$.

   (e) The m.g.f. uniquely defines a distribution.

3. **MGF of Multivariate Normal Variate** If $X \sim N_p(\mu, \Sigma)$ then

   $$M(t) = \exp \left\{ \mu't + \frac{1}{2}t'\Sigma t \right\}.$$

   **Proof** Note that $X = \mu + \Sigma^{\frac{1}{2}}Z$ where $Z \sim N_p(0, I_p)$. Hence

   $$M_X(t) = \exp \{\mu't\} M_Z(\sqrt{\Sigma} t) = \exp \{\mu't\} M_Z(\sqrt{\Sigma} t).$$

   MGF.3
Now,

\[ M_Z(t) = \prod_{i=1}^{p} M_{Z_i}(t_i) \text{ (why?)} \]

\[ = \prod_{i=1}^{p} \exp \left\{ \frac{1}{2} t_i^2 \right\} = \exp \left\{ \frac{1}{2} \sum_{i=1}^{p} t_i^2 \right\} \]

\[ = \exp \left\{ \frac{1}{2} t' t \right\}. \]

The result follows.