E. Multi-Way Fixed-Effects ANOVA Model

1 Balanced Three-Way Fixed-Effects ANOVA Model

Completely crossed three-way design has row factor A at a levels, column factor B at b levels, and depth factor C at c levels where n (replicates) experimental units are assigned to each of the abc cells for a total of \( N = abc \) units. The data are denoted as \( y_{ijk\ell} \) for the \( \ell^{th} \) observation in the \((i,j,k)^{th}\) cell.

1.1 Fixed-Effects Model

\[
y_{ijk\ell} = \mu + \alpha_i + \beta_j + \gamma_k + (\alpha\beta)_{ij} + (\alpha\gamma)_{ik} + (\beta\gamma)_{jk} + (\alpha\beta\gamma)_{ijk} + \varepsilon_{ijk\ell} \tag{E.1}
\]

**Parameters:** \( \mu \) = grand mean; \( \alpha_i, \beta_j, \gamma_k \) are main effects of A, B, and C, respectively; \( (\alpha\beta)_{ij}, (\alpha\gamma)_{ik}, \text{ and } (\beta\gamma)_{jk} \) are A \( \times \) B, A \( \times \) C, and B \( \times \) C two-factor interactions respectively; \( (\alpha\beta\gamma)_{ijk} \) is the three-factor interaction.

**Side Conditions:**

(i) \( \sum_i \alpha_i = \sum_j \beta_j = \sum_k \gamma_k = 0 \).

(ii) \( \sum_i (\alpha\beta)_{ij} = \sum_k (\beta\gamma)_{jk} = 0, \forall j; \sum_j (\alpha\beta)_{ij} = \sum_k (\alpha\gamma)_{ik} = 0, \forall i; \sum_i (\alpha\gamma)_{ik} = \sum_j (\beta\gamma)_{jk} = 0, \forall k. \)

(iii) \( \sum_i (\alpha\beta\gamma)_{ijk} = 0, \forall j \forall k; \sum_j (\alpha\beta\gamma)_{ijk} = 0, \forall i \forall k; \sum_k (\alpha\beta\gamma)_{ijk} = 0, \forall i \forall j. \)

**Assumptions:** \( \varepsilon_{ijk\ell} \overset{i.i.d.}{\sim} N(0, \sigma^2) \). From above, the following assumptions, in descending order of importance, are required:

1. **Randomness**, and **additivity** (of model parameters).
2. **Homogeneous variances**.
3. **Normality** (of error term and hence of observations).

1.2 Sums of Squares

1. **Total Sum of Squares.**

\[
SS_T = \sum_i \sum_j \sum_k \sum_\ell (y_{ijk\ell} - \bar{y}_{...})^2 = \sum_i \sum_j \sum_k \sum_\ell y_{ijk\ell}^2 - \frac{T^2}{N}.
\]
2. Cells Sum of Squares.

\[ SS_{\text{cells}} = n \sum_{i} \sum_{j} \sum_{k} (\bar{y}_{ijk} - \bar{y}_{...})^2 = \sum_{i} \sum_{j} \sum_{k} T_{ijk}^2 - \frac{T^2}{N}. \]

3. Error Sum of Squares.

\[ SS_{E} = \sum_{i} \sum_{j} \sum_{k} \sum_{\ell} (y_{ijk\ell} - \bar{y}_{ijk})^2 = \sum_{i} \sum_{j} \sum_{k} \sum_{\ell} y_{ijk\ell}^2 - \sum_{i} \sum_{j} \sum_{k} T_{ijk} \frac{2}{n}. \]

4. Main-Effect Sums of Squares.

\[ SS_A = bcn \sum_{i} (\bar{y}_{i...} - \bar{y}_{...})^2 = \sum_{i} T_{i...}^2 - \frac{T^2}{N}, \]

\[ SS_B = acn \sum_{j} (\bar{y}_{.j..} - \bar{y}_{...})^2 = \sum_{j} T_{.j..}^2 - \frac{T^2}{N}, \]

\[ SS_C = abn \sum_{k} (\bar{y}_{..k.} - \bar{y}_{...})^2 = \sum_{k} T_{..k.}^2 - \frac{T^2}{N}. \]

5. Two-Factor-Interaction Sums of Squares.

\[ SS_{A \times B} = cn \sum_{i} \sum_{j} (\bar{y}_{ij..} - \bar{y}_{i...} - \bar{y}_{.j..} + \bar{y}_{...})^2 = \left( \sum_{i} \sum_{j} \frac{T_{ij..}^2}{cn} - \frac{T^2}{N} \right) - SS_A - SS_B, \]

\[ SS_{A \times C} = bn \sum_{i} \sum_{k} (\bar{y}_{i.k.} - \bar{y}_{i...} - \bar{y}_{.k.} + \bar{y}_{...})^2 = \left( \sum_{i} \sum_{k} \frac{T_{i.k.}^2}{bn} - \frac{T^2}{N} \right) - SS_A - SS_C, \]

\[ SS_{B \times C} = an \sum_{j} \sum_{k} (\bar{y}_{.jk.} - \bar{y}_{.j..} - \bar{y}_{..k.} + \bar{y}_{...})^2 = \left( \sum_{j} \sum_{k} \frac{T_{.jk.}^2}{an} - \frac{T^2}{N} \right) - SS_B - SS_C. \]

6. Three-factor-interaction Sum of Squares.

\[ SS_{A \times B \times C} = n \sum_{i} \sum_{j} \sum_{k} (\bar{y}_{ijk} - \bar{y}_{ij.} - \bar{y}_{i.k.} - \bar{y}_{i..} - \bar{y}_{.j.} - \bar{y}_{..k.} + \bar{y}_{...})^2 = SS_{\text{cells}} - SS_A - SS_B - SS_C - SS_{A \times B} - SS_{A \times C} - SS_{B \times C}. \]

1.3 Theorems and ANOVA Table

1. \( SS_T = SS_{\text{cells}} + SS_E. \)

2. \( SS_{\text{cells}} = SS_A + SS_B + SS_C + SS_{A \times B} + SS_{A \times C} + SS_{B \times C} + SS_{A \times B \times C}. \)
3. $SS_E/\sigma^2 \sim \chi^2(abc(n - 1))$.

4. $E(\text{MS}_E) = \sigma^2$, where $\text{MS}_E = SS_E/[abc(n - 1)]$.

5. The ANOVA table:

<table>
<thead>
<tr>
<th>Source</th>
<th>SS</th>
<th>DF</th>
<th>MS</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$SS_A$</td>
<td>$a - 1$</td>
<td>$MS_A$</td>
</tr>
<tr>
<td>B</td>
<td>$SS_B$</td>
<td>$b - 1$</td>
<td>$MS_B$</td>
</tr>
<tr>
<td>C</td>
<td>$SS_C$</td>
<td>$c - 1$</td>
<td>$MS_C$</td>
</tr>
<tr>
<td>$A \times B$</td>
<td>$SS_{A\times B}$</td>
<td>$(a - 1)(b - 1)$</td>
<td>$MS_{A\times B}$</td>
</tr>
<tr>
<td>$A \times C$</td>
<td>$SS_{A\times C}$</td>
<td>$(a - 1)(c - 1)$</td>
<td>$MS_{A\times C}$</td>
</tr>
<tr>
<td>$B \times C$</td>
<td>$SS_{B\times C}$</td>
<td>$(b - 1)(c - 1)$</td>
<td>$MS_{B\times C}$</td>
</tr>
<tr>
<td>$A \times B \times C$</td>
<td>$SS_{A\times B\times C}$</td>
<td>$(a - 1)(b - 1)(c - 1)$</td>
<td>$MS_{A\times B\times C}$</td>
</tr>
<tr>
<td>Error</td>
<td>$SS_E$</td>
<td>$abc(n - 1)$</td>
<td>$MS_E$</td>
</tr>
<tr>
<td>Total</td>
<td>$SS_T$</td>
<td>$abcn - 1$</td>
<td></td>
</tr>
</tbody>
</table>

1.4 Rules for the Construction of Expected Mean Squares in Fixed-Effects Models

with the following notation/configuration for a fixed-effects model

for any number of completely crossed factors $A, B, C, D, \cdots$ at $a, b, c, d, \cdots$ levels, respectively with $n$ replicates at each cells.

Consider $u$, a combination of the symbols $A, B, C, D, \cdots$. For instance, $u = AC$ denotes $A \times C$ term. Now,

$$E(\text{MS}_u) = \sigma^2 + c_u k_u^2 \quad \text{(E.2)}$$

where

$$c_u = n \times \text{(product of lower-case letters of terms not in } u)$$

$$k_u^2 = \frac{\text{SS of all effects corresponding to } u}{df_u}.$$ 

1.4.1 Example for Three-Way Fixed-Effects Model

Factors $A, B, C$ with $a, b, c$ respective levels and with $\alpha, \beta, \gamma$ corresponding effects.

- $u = B$, $c_u = nac$, $k_u^2 = \sum_j \beta_j^2/(b - 1)$.
- $u = AC$, $c_u = nb$, $k_u^2 = \sum_i \sum_k (\alpha\gamma_{ik})^2 /[(a - 1)(c - 1)]$.
- $u = ABC$, $c_u = n$, $k_u^2 = \sum_i \sum_j \sum_k (\alpha\beta\gamma)_{ijk}^2 /[(a - 1)(b - 1)(c - 1)]$. 

E.3
1.5 Recommended Analysis

Run a test on complete null hypothesis on cell means first followed by test on three-way interaction effects if cell means differ significantly. Insignificant three-way interaction effects leads to testing two-way interactions. A test on the main effects of a particular factor makes sense only if the factor is not involved in any significant interaction. Tukey-analysis of the main effects of a factor makes sense only if it’s not involved in significant interactions. Otherwise, simple-effect analysis should be conducted.

1.6 Drink Bottler Example

This example is from Montgomery. A soft drink bottler is interested in the effect of percent carbonation ($A$), operating pressure in the filter ($B$), and line speed ($C$) on the volume of carbonated beverage ($y$) packaged in each bottle. Three levels of carbonation ($a = 3$), two levels of pressure ($b = 2$), and two levels of speed ($c = 2$) are selected, and a factorial experiment with two replicates ($n = 2$) is conducted. The run order is completely randomized. The following data were collected:

<table>
<thead>
<tr>
<th>carbonation ($A$)</th>
<th>pressure ($B$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>25psi</td>
</tr>
<tr>
<td></td>
<td>speed ($C$)</td>
</tr>
<tr>
<td></td>
<td>200bpm</td>
</tr>
<tr>
<td>10%</td>
<td>7 9 9 10</td>
</tr>
<tr>
<td>12%</td>
<td>10 11 12 11</td>
</tr>
<tr>
<td>14%</td>
<td>15 14 17 16</td>
</tr>
</tbody>
</table>