G. Nested Designs

1 Introduction.

1.1 Definition (Nested Factors)
When each level of one factor B is associated with one and only one level of another factor A, we say that B is nested within factor A.

1.2 Notation
Use $B_\omega A$ to denote that B is nested within factor A.

1.3 Example
PVPS Example (see “Two-Way Fixed-Effects ANOVA Model” on pages D.2 & D.3.).

2 Two-Way Nested Designs (Balanced Cases)

2.1 Definition
$y_{ij\ell} = \mu + \alpha_i + \beta_{j(i)} + \epsilon_{ij\ell}, i = 1, \ldots, a, j = 1, \ldots, b, \ell = 1, \ldots, n,$ where
$\mu =$ overall population mean
$\alpha_i =$ $i$th effect of factor A
$\beta_{j(i)} =$ $j$th level effect of factor B nested within the $i$th level of factor A
$\epsilon_{ij\ell} =$ error ($N = abn$ i.i.d. $N(0, \sigma^2)$).

2.2 Assumptions
1. Fixed effects case: $\sum_{i=1}^{a} \alpha_i = 0$ (A fixed); $\sum_{j=1}^{b} \beta_{j(i)} = 0$ for each $i = 1, \ldots, a$ (B is fixed within each level of A).

2. Random effects case: $\{\alpha_1, \ldots, \alpha_a\}$ form a random sample of $N(0, \sigma_a^2)$ (A random); $\{\beta_{1(i)}, \ldots, \beta_{b(i)}\}$ form a random sample of $N(0, \sigma_b^2)$ for each $i = 1, \ldots, a$ (B is random within each level of A); all $ab + abn$ r.v.’s are independent.

3. Mixed effects model: A is fixed; B is random: $\sum_{i=1}^{a} \alpha_i = 0$; $\{\beta_{1(i)}, \ldots, \beta_{b(i)}\}$ form a random sample of $N(0, \sigma_b^2)$ for each $i = 1, \ldots, a$; all $ab + abn$ r.v.’s are independent.

4. (Rare, not considered here): A is random; B is fixed.
2.3 Notation

\( \bar{y}_{ij} \) = average response for the \( j \)th level of B nested within the \( i \)th level of A.

\( \bar{y}_{i..} = \sum_{j=1}^{b} \sum_{\ell=1}^{n} \frac{y_{ij\ell}}{bn} = \) average response for the \( i \)th level of A.

\( \bar{y}_{...} = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{\ell=1}^{n} \frac{y_{ij\ell}}{abn} = \) overall sample mean.

2.4 Sums of Squares, Mean Squares, and Computational Formulas

1. \( SS_T \) = \( \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{\ell=1}^{n} (y_{ij\ell} - \bar{y}_{...})^2 = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{\ell=1}^{n} y_{ij\ell}^2 - \bar{T}_{..}^2 / abn \)

2. \( SS_E \) = \( \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{\ell=1}^{n} (y_{ij\ell} - \bar{y}_{ij.})^2 ; MS_E = SS_E / ab(n - 1) \).

3. \( SS_A \) = \( \sum_{i=1}^{a} nb(\bar{y}_{i..} - \bar{y}_{...})^2 = \sum_{i=1}^{a} T_{i..}^2 / nb - \bar{T}_{..}^2 / abn ; MS_A = SS_A / (a - 1) \).

4. \( SS_{B(A)} \) = \( \sum_{i=1}^{a} \sum_{j=1}^{b} n(\bar{y}_{ij.} - \bar{y}_{...})^2 = \sum_{i=1}^{a} \sum_{j=1}^{b} T_{ij.}^2 / n - \sum_{i=1}^{a} T_{i..}^2 / nb ; MS_{B(A)} = SS_{B(A)}/a(b - 1) \).

2.5 Theorems

1. \( SS_T = SS_A + SS_{B(A)} + SS_E \).

2. Regardless of which model is correct (in § G2.2):

   \( \alpha \). \( SS_E / \sigma^2 \sim \chi^2(ab(n - 1)) \).

   \( \beta \). \( E[MS_E] = \sigma^2 \).

   **Proof:** Note that \( SS_E = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{\ell=1}^{n} (\varepsilon_{ij\ell} - \bar{\varepsilon}_{ij.})^2 \).

2.6 Hypotheses of interest

1. Test A:

   \( H_0^A : \alpha_i = 0, \forall i = 1, \ldots, a \) (fixed)

   \( H_0^A : \sigma_a^2 = 0 \) (random)

2. Test B\( \omega \)A:

   \( H_0^B : \beta_{j(i)} = 0, \forall i \) and \( j \) (fixed)

   \( H_0^B : \sigma_b^2 = 0 \) (random)

2.7 Definition

If A is fixed: \( k^2_a = \frac{1}{a-1} \sum_{i=1}^{a} \alpha_i^2 \).

If B is fixed: \( k^2_b = \frac{1}{a(b-1)} \sum_{i=1}^{a} \sum_{j=1}^{b} \beta_{j(i)}^2 \).
2.8 ANOVA Table

<table>
<thead>
<tr>
<th>Source</th>
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<th>DF</th>
<th>MS</th>
<th>E[MS]</th>
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<tr>
<td>A</td>
<td>$SS_A$</td>
<td>$a-1$</td>
<td>$MS_A$</td>
<td>$\sigma^2 + nbk_a^2$</td>
</tr>
<tr>
<td>B:ωA</td>
<td>$SS_{B(A)}$</td>
<td>$(a(b-1))$</td>
<td>$MS_{B(A)}$</td>
<td>$\sigma^2 + n\sigma_b^2$</td>
</tr>
<tr>
<td>Residual</td>
<td>$SS_E$</td>
<td>$ab(n-1)$</td>
<td>$MS_E$</td>
<td>$\sigma^2$</td>
</tr>
<tr>
<td>Total</td>
<td>$SS_T$</td>
<td>$abn-1$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2.9 More Theorems

1. An $\alpha$-level test of $H_0^B: \sigma_b^2 = 0$ or $H_0^B: k_b^2 = 0$ is

$$F = \frac{MS_{B(A)}}{MS_E} > F_{\alpha;a(b-1),ab(n-1)}$$

regardless of which model is correct.

2. An $\alpha$-level test of $H_0^A: \alpha_i = 0, \forall i = 1, \ldots, a$ is

$$F = \frac{MS_A}{MS_E} > F_{\alpha;a-1,ab(n-1)}$$

in the fixed effects models.

3. An $\alpha$-level test of $H_0^A: \alpha_i = 0, \forall i = 1, \ldots, a$ is

$$F = \frac{MS_A}{MS_{B(A)}} > F_{\alpha;a-1,a(b-1)}$$

in the mixed effects model.

4. An $\alpha$-level test of $H_0^A: \sigma_a^2 = 0$ is

$$F = \frac{MS_A}{MS_{B(A)}} > F_{\alpha;a-1,a(b-1)}$$

in the random effects model.

5. For the random effects model:

$$\text{Var}(y_{ij\ell}) = \sigma^2 + \sigma_a^2 + \sigma_b^2 = \sigma_{\text{tot}}^2$$

and the variance components are estimated as:

<table>
<thead>
<tr>
<th>variance component</th>
<th>estimate</th>
<th>proportion</th>
</tr>
</thead>
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<tr>
<td>$\sigma^2$</td>
<td>$MS_E$</td>
<td>$\frac{\hat{\sigma}^2}{\sigma_{\text{tot}}^2}$</td>
</tr>
<tr>
<td>$\sigma_b^2$</td>
<td>$\frac{(MS_{B(A)} - MS_E)}{n}$</td>
<td>$\frac{\hat{\sigma}<em>b^2}{\sigma</em>{\text{tot}}^2}$</td>
</tr>
<tr>
<td>$\sigma_a^2$</td>
<td>$\frac{(MS_A - MS_{B(A)})}{nb}$</td>
<td>$\frac{\hat{\sigma}<em>a^2}{\sigma</em>{\text{tot}}^2}$</td>
</tr>
<tr>
<td>$\sigma_{\text{tot}}^2$</td>
<td>$\hat{\sigma}_{\text{tot}}^2 = \sigma^2 + \hat{\sigma}_b^2 + \hat{\sigma}_a^2$</td>
<td>$\frac{\hat{\sigma}<em>{\text{tot}}^2}{\sigma</em>{\text{tot}}^2}$</td>
</tr>
</tbody>
</table>
2.10 PVPS Example, revisited

Consider again the “PVPS Example” (pp. D.2 & D.3) with

\[ a = 2 \text{ schools} \quad b = 3 \text{ classes/school} \]
\[ n = 4 \text{ students per class/school} \]
\[ y_{ijℓ} = \text{reading achievement score} \]

Assume the following pseudo data:

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<th>S2</th>
<th></th>
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</thead>
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<td>C3</td>
<td>C4</td>
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<td>10</td>
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<td>17</td>
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<td></td>
<td>48</td>
<td>68</td>
<td>50</td>
<td>75</td>
</tr>
<tr>
<td></td>
<td>166</td>
<td>224</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ T_{ij} = \begin{array}{cccc}
12 & 15 & 10 & 18 \\
13 & 19 & 12 & 20 \\
11 & 17 & 11 & 21 \\
12 & 17 & 17 & 16 \\
\end{array} \]

\[ T_{i..} = \begin{array}{c}
48 \\
68 \\
50 \\
75 \\
\end{array} \]

\[ T_{..} = 390 \]

\[ SS_T = \sum_{i=1}^{2} \sum_{j=1}^{3} \sum_{ℓ=1}^{4} y_{ijℓ}^2 - T_{ij}^2/24 = 6654 - 390^2/24 = 6654 - 6337.5 = 316.5 \]

\[ SS_A = \sum_{i=1}^{2} T_{i.}^2/12 - T_{ij}^2/24 = (166^2 + 224^2)/12 - 6337.5 = 6477.67 - 6337.5 = 140.17 \]

\[ SS_{B(A)} = \sum_{i=1}^{2} \sum_{j=1}^{3} T_{ij}^2/4 - \sum_{i=1}^{2} T_{i.}^2/12 = (48^2 + 68^2 + 50^2 + 75^2 + 66^2 + 83^2)/4 - 6477.67 = 96.83 \]

\[ SS_E = SS_T - SS_A - SS_{B(A)} = 79.50 \]

AOV Table

<table>
<thead>
<tr>
<th>Source</th>
<th>SS</th>
<th>DF</th>
<th>E(MS)</th>
</tr>
</thead>
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<tr>
<td></td>
<td></td>
<td></td>
<td>fixed</td>
</tr>
<tr>
<td>School</td>
<td>140.17</td>
<td>1</td>
<td>(σ^2 + 12 \sum_{i=1}^{2} α_i^2)</td>
</tr>
<tr>
<td>Class(School)</td>
<td>96.83</td>
<td>4</td>
<td>(σ^2 + 2 \sum_{i=1}^{2} \sum_{j=1}^{3} β_{ij}^2)</td>
</tr>
<tr>
<td>Residual</td>
<td>79.50</td>
<td>18</td>
<td>(σ^2)</td>
</tr>
<tr>
<td>Total</td>
<td>316.5</td>
<td>23</td>
<td></td>
</tr>
</tbody>
</table>

1. Case 1: Fixed effects
   - School: \(MS_A / MS_E = 31.73 > 4.41 = F_{0.05;1,18}\) significant
   - Class: \(MS_{B(A)} / MS_E = 5.48 > 2.93 = F_{0.05;4,18}\) significant.

2. Case 2: Random effects
   - School: \(MS_A / MS_{B(A)} = 5.79 \neq 7.71 = F_{0.05;1,4}\) insignificant.
   - Class: Same as case 1.

3. Case 3: Mixed effects
   - School: same as case 2.
   - Class: same as case 1.
2.11 Inferences About the Fixed Factor Means

For the fixed-effects cases, Levene’s test of $H_0: \alpha_i = \alpha'_i$ for $i = 1, \ldots, a$ as follows.

Let $L = \sum_{i=1}^a h_i \mu_i$ be a contrast ($\sum_{i=1}^a h_i = 0$) and $\hat{L} = \sum_{i=1}^a h_i \bar{y}_{..}$.

$$\text{std.err}(\hat{L}) = \sqrt{\frac{\sum_{i=1}^a h_i^2/(nb)}{\sum_{i=1}^n h_i^2/(nb)} \times MS_E}.$$ 

For the stage 2 tests of $H_0: \alpha_i = \alpha'_i$, the following $\alpha$-level procedures are used in the $\alpha$-level protected LSD tests of $H_0^A: \alpha_1 = \cdots = \alpha_a = 0$:

$$\begin{cases} \frac{|\bar{y}_{i..} - \bar{y}_{i'..}|}{\sqrt{2MS_E/nb}} > t_{\alpha/2; ab(n-1)}, & \text{if fixed effects} \\ \frac{|\bar{y}_{i..} - \bar{y}_{i'..}|}{\sqrt{2MS_B(A)/nb}} > t_{\alpha/2; a(b-1)}, & \text{if random effects} \end{cases}$$

In the fixed-effects model stage 2 tests of $H_0: \mu_{ij} = \mu_{ij'} \iff H_0: \beta_{j(i)} = \beta_{j'(i)}$, comparing the nested $j$ and $j'$ levels of $B$ within the $i$th level of $A$, are wanted. An $\alpha$-level test of $H_0: \mu_{ij} = \mu_{ij'}$ is

$$\frac{|\bar{y}_{ij} - \bar{y}_{ij'}|}{\sqrt{2MS_E/n}} > t_{\alpha/2; ab(n-1)}.$$

For the fixed-effects cases Levene’s test of $H_0: \sigma_{ij}^2 = \sigma^2$ for all $i = 1, \ldots, a$, and $j = 1, \ldots, b$ can be obtained by inputting the $k = ab$ groups of data into AOV, where the data for the $(i, j)$th group is $y_{ij1}, \ldots, y_{ijn}$ with $s_{ij}^2 = \sum_{\ell=1}^n (y_{ij\ell} - \bar{y}_{ij})^2/(n-1)$ for $i = 1, \ldots, a$, and $j = 1, \ldots, b$. If L-prob. is insignificant at $\alpha = .25$, then the stage 2 tests of $H_0: \alpha_i = \alpha'_i$ and $H_0: \mu_{ij} = \mu_{ij'}$ using $MS_E$ are valid; otherwise these procedures are not valid. For the mixed model, Levene’s procedure is not valid. The test for appropriateness of the use of $MS_B(A)$ as the stage 2 error-term is open for these cases.

2.12 Spray Example

We consider a study of three different sprays used on trees. Each of the three sprays was applied to four trees. After one week the concentration of nitrogen was measured in each of six leaves picked in a random way from each tree. Here the experimental units are the 12 trees; we consider them to have been chosen at random from a large population of trees. The leaves thus form four subsamples, each consisting of six leaves from a large population of leaves on the particular tree. Note that in this experiment 12 trees were sprayed and 72 leaves were picked. Each measurement of nitrogen concentration is denoted by $y_{ijk}$, where $i$, the number of the treatment, runs from 1 to $a$; $j$, the number of the experimental unit for the $i$th treatment, runs from 1 to $b$; and $k$, the observation number from the $j$th experimental unit on the $i$th treatment, runs from 1 to $n$. $y_{ijk}$: the nitrogen concentration of the $k$th leaf from the $j$th tree that received spray $i$. 

G.5
2.13 Proofs of Theorems in §G.2.9

1. **Case 1:** Fixed effects model.

   **Claim #1:** \( SS_{B(A)} / \sigma^2 \sim \chi^2(a(b - 1), \lambda) \), where
   \[
   SS_{B(A)} / \sigma^2 = \sum_{i=1}^{a} \sum_{j=1}^{b} n / \sigma^2 (\bar{y}_{ij} - \bar{y}_i)^2 = \sum_{i=1}^{a} n / \sigma^2 \sum_{j=1}^{b} (\beta_{ji} + \bar{z}_{ij} - \bar{z}_i)^2 = \sum_{i=1}^{a} n / \sigma^2 \sum_{j=1}^{b} (X_{ij} - X_i)^2,
   \]
   where \( \beta_{ji} + \bar{z}_{ij} = X_{ij} - X_i \). Since \( \beta_{ji} \) are independent, \( X_{ij} \sim N(\beta_{ji}, \sigma^2/n) \) and \( X_i = \sum_{j=1}^{b} X_{ij} / b \). So, \( Q_i / \sigma^2 = n / \sigma^2 \sum_{j=1}^{b} (X_{ij} - X_i)^2 \sim \chi^2(b - 1, \lambda_i) \), \( \lambda_i = n / \sigma^2 \sum_{j=1}^{b} \beta_{ji}^2 \) for \( i = 1, \ldots, a \). Consequently, \( SS_{B(A)} / \sigma^2 \sim \chi^2(a(b - 1), \lambda) = \sum_{i=1}^{a} \lambda_i \).

   **Claim #2:** \( SS_{B(A)} / \sigma^2 = n / \sigma^2 \sum_{i=1}^{a} \sum_{j=1}^{b} (\beta_{ji} + \bar{z}_{ij} - \bar{z}_i)^2 \) and \( SS_E / \sigma^2 = 1 / \sigma^2 \sum_{i=1}^{a} \sum_{j=1}^{b} (\varepsilon_{ij} - \bar{z}_{ij} - \bar{z}_i)^2 \) are independent:
   \( SS_{B(A)} = f_1(\bar{z}_{ij} - \bar{z}_i) \) and \( SS_E = f_2(\varepsilon_{ij} - \bar{z}_{ij}) \). \( SS_{B(A)} \) and \( SS_E \) are independent since \( \bar{z}_{ij} - \bar{z}_i \) and \( \varepsilon_{ij} - \bar{z}_{ij} \) are orthogonal classes of contrasts when the \( \varepsilon_{ij} \)'s are i.i.d. (why?)

The fact that \( MS_{B(A)} / MS_E \) is non-central \( F(a(b - 1), ab(n - 1); n / \sigma^2 \sum_{i=1}^{a} \sum_{j=1}^{b} \beta_{ji}^2) \)

___

**TABLE:**

<table>
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<tr>
<th>SPRAY</th>
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<th>3</th>
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<td>11.44</td>
<td>15.31</td>
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</tbody>
</table>

See computer example online.
follows from the above claims. It follows that $MS_{B(A)}/MS_E$ is $F(a(b - 1), ab(n - 1))$ when $H_0^B : \beta_{j(i)} = 0, \forall i, \forall j$.

**Case 2:** Random effects model.

- **Claim #3:** $SS_{B(A)}/(\sigma^2 + n\sigma_b^2) \sim \chi^2(a(b - 1))$ (proof: exercise).
- **Claim #4:** $SS_{B(A)}$ and $SS_E$ are independent under the random effects model (proof: exercise).

It follows from claims #3 and #4 that

$$\frac{MS_{B(A)}}{MS_E} \cdot \frac{\sigma^2}{\sigma^2 + n\sigma_b^2} \sim F(a(b - 1), ab(n - 1))$$

and that $MS_{B(A)}/MS_E \sim F(a(b - 1), ab(n - 1))$ when $H_0^B : \sigma_b^2 = 0$ is true.

**Case 3:** Mixed effects model.

The result for the mixed effects model holds because this is identical to the distributional properties for the random effects cases.

2. The distributions of $MS_A$ and $MS_E$ under the nested model are exactly equivalent to the distributions of $MS_A$ and $MS_E$ under the balanced two-way fixed model (completely crossed).

3. $SS_A = nb \sum_{i=1}^{a}(\bar{y}_{i.} - \bar{y}_{..})^2 = nb \sum_{i=1}^{a}[(\alpha_i + \beta_{j(i)} + \bar{z}_{i.}) - (\bar{\beta}_{..} + \bar{z}_{..})]^2$. $\bar{y}_{i.}$’s are independent $N(\mu + \alpha_i, \sigma_b^2/b + \sigma^2/bn)$. So,

$$\sum_{i=1}^{a} \frac{(\bar{y}_{i.} - \bar{y}_{..})^2}{\sigma_b^2/b + \sigma^2/bn} = \frac{SS_A}{\sigma^2 + n\sigma_b^2} \sim \chi^2(a - 1, \frac{nb}{\sigma^2 + n\sigma_b^2} \sum_{i=1}^{a} \alpha_i^2) \equiv \chi^2(a - 1, \frac{nb(a - 1)k_a^2}{\sigma^2 + n\sigma_b^2})$$

when $B(A)$ is random, we have $SS_{B(A)}/(\sigma^2 + n\sigma_b^2) \sim \chi^2(a(b - 1))$ (claim #3 above).

Finally, $SS_A = f_1(\bar{\beta}_{..} - \bar{\beta}_{..}, \bar{z}_{..} - \bar{z}_{..})$ and $SS_{B(A)} = f_2(\beta_{j(i)} - \bar{\beta}_{..}, \bar{z}_{ij} - \bar{z}_{..})$. It can be shown under the mixed effects model that $\bar{\beta}_{..} - \bar{\beta}_{..}, \bar{z}_{..} - \bar{z}_{..}, \beta_{j(i)} - \bar{\beta}_{..}$, and $\bar{z}_{ij} - \bar{z}_{..}$ are four orthogonal classes of r.v.’s. Hence, $SS_A$ and $SS_{B(A)}$ are independent. Hence,

$$\frac{MS_A}{MS_{B(A)}} \sim F(a - 1, a(b - 1); \frac{nb}{\sigma^2 + n\sigma_b^2} \sum_{i=1}^{a} \alpha_i^2)$$

and

$$\frac{MS_A}{MS_{B(A)}} \sim F(a - 1, a(b - 1)) \text{ under } H_0^A : \alpha_1 = \cdots = \alpha_a = 0.$$